Haar wavelet collocation method for the numerical solution of boundary layer fluid flow problems

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Based on Haar wavelets an efficient numerical method is proposed for the numerical solution of system of coupled Ordinary Differential Equations (ODEs) related to the natural convection boundary layer fluid flow problems with high Prandtl number (Pr). The numerical study of these flow models is necessary as the existing literature is more focused on the flow problems with small values of Pr. In this work, the problem of natural convection which consists of coupled nonlinear ODEs is solved simultaneously. The ODEs are obtained from the Navier Stokes equations through the similarity transformations. The effects of variation of Pr on heat transfer are investigated. Performance of the Haar Wavelets Collocation Method (HWCM) is compared with the finite difference method (FDM), Runge–Kutta Method (RKM), homotopy analysis method (HAM) and exact solution for the last problem. More accurate solutions are obtained by wavelets decomposition in the form of a multi-resolution analysis of the function which represents solution of the given problems. Through this analysis the solution is found on the coarse grid points and then refined towards higher accuracy by increasing the level of the Haar wavelets. Neumann’s boundary conditions which are problematic for most of the numerical methods are automatically coped with. A distinctive feature of the proposed method is its simple applicability for a variety of boundary conditions. Efficiency analysis of HWCM versus RKM is performed using Timing command in Mathematica software. A brief convergence analysis of the proposed method is given. Numerical tests are performed to test the applicability, efficiency and accuracy of the method.

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1. Introduction

High Prandtl number fluids are frequently encountered in industry such as fluids used as a heat sink in electrical transformer with Pr = 47100 at 273 K, hydrocarbon polymers or silicones used in some chemical processes and substances such as glycerine with Pr = 8470 at 273 K. Geological flows involve fluids with very large Pr (Pr = 1000 for magmas and at least 1023 for the earth mantle), engine oil (Pr = 100–40,000), starting plumes rising through viscous oils (Pr = 10^4), etc [21,35]. The governing partial differential equations in cartesian co-ordinates which consist of continuity, momentum and energy for these flow models are converted to ODEs through similarity transformations. Due to scarcity of efficient numerical methods, the resulting systems of simultaneous coupled nonlinear ODEs are often very challenging for the existing methods in the case of large value of Pr. Efficient numerical methods are needed for numerical solution of highly nonlinear system of ODEs where the analytical solutions appear infeasible. To the best of our knowledge only a few papers are dedicated to the numerical solution of these types of ODEs. In [6,41] the authors discussed the existence and uniqueness of solution to the second-order systems. The numerical methods include FDM and adjoint operator methods [30], reproducing kernel space [10], variational iteration method (VIM) [28], third-degree B-spline [3], sinc-collocation method [7], Chebychev finite difference method [36]. Among the numerical methods, RKM with adaptive or uniform step sizes coupled with nonlinear shooting method [30] is mostly used for the numerical solution of the coupled system of ODEs arising from the nonlinear fluid models through the similarity transformations. This is done in two stages:

(i) Shooting method is used to integrate the boundary value problems as an initial value problem with guesses for the unknown initial values,
(ii) Initial value problem is reduced to a system of first-order ODEs and then solved by RKM.

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The shooting method can be very successful for solving simple problems such as the projectile problem. However, its success depends on a number of factors [12]. The most important of which is the stability of the corresponding initial value problem that must be solved at each iteration. Unfortunately, for many stable boundary value problems the corresponding initial value problems (beginning from either endpoint and integrating towards the other endpoint) are insufficiently stable for shooting to succeed. So, shooting methods are not always computationally suitable for the whole range of practical boundary value problems, particularly those on a very long or infinite intervals. Hence, shooting seems to offer less hope for some of the practical engineering problems [23,29].

The other major alternative to the numerical methods are the approximate methods based on the series solutions. Most prominent of them are differential transform method (DTM) [45], homotopy analysis method (HAM) [27] and homotopy perturbation method (HPM) [17]. These methods have recently been applied to coupled nonlinear ODEs resulting from reducing continuity, momentum and energy equations through similarity transformations. Operational details about these methods can be found in [1,16,8,33,2,42–44] and the references therein. For most of these methods the convergence is not guaranteed and the so called h-curve of the convergence region depends strongly on the magnitudes of the non-dimensional parameters like Pr, Reynolds number (Re), Hartmann number (Hr) etc [8]. The strong influence of these parameters on the size of the valid region is not surprising, considering the fact that it appears in the non-linear term in the governing equation which ultimately makes it more challenging for the asymptotic methods.

This paper proposes HWCM based on the simple Haar wavelets. This approach has the following advantages

(i) HWCM provides accurate solution in the case of flows with high Prandtl numbers, Reynolds numbers, Hartman numbers and Rayleigh numbers,

(ii) Less CPU time is needed in HWCM as the major blocks of HWCM need to be calculated once and are used in the subsequent computations repeatedly,

(iii) Unlike RKM, HWCM performs very well for a boundary value problem defined on a very long interval,

(iv) Contrary to RKM, HWCM does not require conversion of a boundary value problem into initial value problem by using a procedure like shooting and hence the boundary-value problem is not integrated as an initial value problem with guesses for the unknown initial values. This property of HWCM eliminates the possibility of unstable solution due to missing initial condition in the case of RKM,

(v) Unlike RKM the boundary value problem needs not to be reduced into a system of first order ODEs,

(vi) A variety of boundary conditions can be handled with equal ease,

(vii) Simple and direct applicability with no need of other intermediate technique is required.

In recent years the wavelet approach is becoming increasingly popular in the field of numerical approximations. Different types of wavelets and approximating functions have been used in numerical solution of boundary-value problems. Out of these, the Haar wavelets [5] have gained popularity among researchers due to their useful properties. In most of the cases, the beauty of wavelet approximation is overshadowed by computational cost of the algorithm. Haar wavelets are the simplest orthonormal wavelets with a compact support. Refs. [25,26] applied Haar wavelets in solving nonlinear integro-differential equations and partial differential equations. For details one may refer to the Refs. [24,26,38] and [19,18]. Haar wavelets are preferred due to their useful properties such as simple applicability, orthogonality and compact support. Compact support of the Haar-wavelet basis permits straight inclusion of the different types of boundary conditions in the numerical algorithms. Due to the linear and piecewise nature, Haar wavelet basis lacks differentiability and hence the integration approach is used instead of the differentiation for calculation of the coefficients. The attributes of other differentiable wavelets like the wavelets of high order spline basis are overshadowed by the computational cost of the algorithms obtained from these wavelets. For the ease of understanding we define the frequently used symbols in the table under the caption Nomenclature.

The present work is organized as follows. In section 2, Haar wavelets are introduced. HWCM is applied to three types of boundary layer fluid flow problems in section 3. In section 4, we present a brief convergence analysis and in section 5 numerical algorithm is discussed. Section 6 is devoted to viscoelastic fluid with known exact solution. Numerical results are presented in section 7. Conclusions of the work along with the future research are given in section 8.

## 2. Haar wavelets

The one dimensional Haar wavelet family for $x \in (0, 1)$ is defined as

\[
h_i(x) = \begin{cases} 
1, & \text{for } x \in [\alpha, \beta), \\
-1, & \text{for } x \in [\beta, \gamma), \\
0, & \text{elsewhere}, 
\end{cases}
\]

(1) where

\[
\alpha = \frac{k}{m}, \quad \beta = \frac{k + 0.5}{m}, \quad \gamma = \frac{k + 1}{m}
\]

(2)
In the above definition integer \( m = 2^j, j = 0, 1, \ldots, J \), indicates the level of the wavelet and the integer \( k = 0, 1, \ldots, m - 1 \) is the translation parameter. Maximum level of resolution is \( J \). The index \( i \) in Eq. (1) is calculated by using the formula \( i = m + k + 1 \). In the case with minimal values \( m = 1, k = 0 \), we have \( i = 2 \). The maximal value of \( i \) is \( i = 2M = 2^{J+1} \). We consider the collocation points

\[
x_j = j - 0.5 \frac{M}{2M}, j = 1, 2, \ldots, 2M.
\]

For \( i = 1 \), the function \( h_1(x) \) is the scaling function for the family of Haar wavelets defined as

\[
h_1(x) = \begin{cases} 1 & \text{for } x \in [0, 1), \\ 0 & \text{elsewhere}. \end{cases}
\]

For the ease of implementation, we follow the same notations as used in [24] for Haar function and their integrals.

\[
p_{i1}(x) = \int_0^x h_i(x')dx',
\]

\[
p_{i(q+1)}(x) = \int_0^x p_{i(q)}(x')dx', \quad q = 1, 2, \ldots.
\]

These integrals can be evaluated using Eq. (1) and the first three of them are given by

\[
p_{i1}(x) = \begin{cases} x - \alpha & \text{for } x \in [\alpha, \beta), \\ \gamma - x & \text{for } x \in [\beta, \gamma), \\ 0 & \text{elsewhere}, \end{cases}
\]

\[
p_{i2}(x) = \begin{cases} \frac{1}{4m^2}(x - \alpha)^2 & \text{for } x \in [\alpha, \beta), \\ \frac{1}{4m^2} - \frac{1}{2}(\gamma - x)^2 & \text{for } x \in [\beta, \gamma), \\ \frac{1}{4m^2} & \text{for } x \in [\gamma, 1), \\ 0 & \text{elsewhere}, \end{cases}
\]

\[
p_{i3}(x) = \begin{cases} \frac{1}{4m^2}(x - \alpha)^3 & \text{for } x \in [\alpha, \beta), \\ \frac{1}{4m^2}(x - \beta) - \frac{1}{6}(\gamma - x)^3 & \text{for } x \in [\beta, \gamma), \\ \frac{1}{4m^2}(x - \beta) & \text{for } x \in [\gamma, 1), \\ 0 & \text{elsewhere}, \end{cases}
\]

We also introduce the following notation

\[
C_{i1} = \int_0^L p_{i1}(x)dx',
\]

\[
C_{i2} = \int_0^L h_i(x')dx',
\]

where \( L \) is a sufficiently large number.

Any function \( f(x) \) which is square integrable in the interval \([0,1)\) can be expressed as an infinite sum of Haar wavelets as

\[
f(x) = \sum_{i=1}^{\infty} a_i h_i(x),
\]

where \( a_i, i = 1, 2, \ldots \) are the Haar coefficients. The above series terminates at finite terms if \( f(x) \) is a piecewise constant or can be approximated as a piecewise constant during each subinterval. The best way to understand the wavelets is through a multi-resolution analysis. Given a function \( f \in L_2(\mathbb{R}) \) a multi-resolution analysis (MRA) of \( L_2(\mathbb{R}) \) produces a sequence of subspaces \( V_j \) \( j \in \mathbb{Z} \) with the following properties

(i) \( \ldots \subset V_{-1} \subset V_0 \subset \ldots \)

(ii) The spaces \( V_j \) satisfy \( \bigcup V_j \) is dense in \( L_2(\mathbb{R}) \) and \( \bigcap V_j = 0 \).

(iii) If \( f(x) \in V_0, f(2x) = V_j \), i.e. the spaces \( V_j \) are scaled versions of the central space \( V_0 \).

(iv) If \( f(x) \in V_0, f(2x-k) \in V_j \) i.e. all the \( V_j \) are invariant under translation.

(v) There exists \( \Phi \in V_0 \) such that \( \Phi(x-k); k \in \mathbb{Z} \) is a Riesz basis in \( V_0 \).

The space \( V_j \) is used to approximate general functions by defining appropriate projection of these functions onto these spaces. Since the union of all the \( V_j \) is dense in \( L_2(\mathbb{R}) \), so it guarantees that any function in \( L_2(\mathbb{R}) \) can be approximated arbitrarily close by such projections. As an example the space \( V_j \) can be defined like

\[
V_j = W_{j-1} \oplus V_{j-1} = W_{j-2} \oplus V_{j-2} = \cdots = \oplus_{i=j+1}^{\infty} W_j \oplus V_0.
\]

then the scaling function \( h_1(x) \) generates an MRA for the sequence of spaces \( \{V_j; j \in \mathbb{Z}\} \) by translation and dilation as defined in Eqs. (1) and (4). For each \( j \) the space \( W_j \) serves as the orthogonal complement of \( V_j \) in \( V_{j+1} \). The space \( W_j \) includes all the functions in \( V_{j+1} \) that are orthogonal to all those in \( V_j \) under some chosen inner product. The set of functions [13] which forms a basis for the space \( W_j \) are called wavelets.
3. Proposed method for different cases of natural convection boundary layer flow

In this section we discuss the formulation of the wavelets approximation to different boundary layer fluid flow cases namely: (i) natural convection boundary layer flow, (ii) laminar film condensation of a saturated stream on an isothermal vertical plate, (iii) boundary layer flow and heat transfer due to a stretching sheet. These flows are important due to their relevance in many practical problems in science and engineering. Some of the applications of these flow models and methods of solution can be found in [32,37,34,2,42,1,33,43,44,8,16,14,22,4]. We discuss the formulation of the new numerical method for each of the three cases separately in the following sections.

3.1. Natural convection boundary layer flow (NCBLF)

The natural convection flows are caused by the density gradient. The density variation in the fluid is caused by the temperature difference. Such flows occur in the vicinity of external surfaces and within channels in which the fluid flows. These type of flows have numerous applications in geothermal and geophysical engineering. These include extraction of geothermal energy, the migration of moisture in brous insulation, underground disposal of nuclear waste and spreading of chemical pollutants in saturated soil [32,30,14,22,4,11]. The governing partial differential equations which consist of continuity, momentum and energy of two dimensional flow over a vertical flat plate with uniform surface temperature as shown in Fig. 1 are [30]:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{13}
\]

where \( u \) and \( v \) are the stream functions and \( T \) is the temperature. The density variation in the fluid is caused by the temperature gradient.

\[
\frac{\partial u}{\partial x} + \frac{\partial T}{\partial y} = \frac{\partial^2 u}{\partial y^2} + g \beta \left( \frac{T - T_m}{T_w - T_m} \right), \tag{14}
\]

\[
\frac{\partial u}{\partial x} + \frac{\partial T}{\partial y} = \frac{\partial^2 T}{\partial y^2}. \tag{15}
\]

subject to the boundary conditions

\[
y = 0 : u = 0, \quad \frac{\partial y}{\partial x} = \frac{\partial T}{\partial y} = 0, \quad T = T_m(x), \quad y \to \infty : u = 0, \quad T = T_m, \tag{16}
\]

where \( x, y \) are the co-ordinates measured parallel and perpendicular to the plate. Define the stream functions \( u = \frac{\partial N}{\partial x} \) and \( v = \frac{\partial N}{\partial y} \) and introducing the following transformations [30]

\[
\eta = y \left[ \frac{g \beta}{C_0} \right]^\frac{1}{2},
\]

\[
\theta(\eta) = \frac{T - T_m}{T_w - T_m},
\]

\[
\psi = 4\nu \left[ \frac{g \beta}{C_0} \right]^\frac{1}{2} f(\eta).
\]

Using these transformations the above equations (13)–(15) reduce to the following equations:

\[
f''(\eta) + 3ff'(\eta) - 2f'^2(\eta) + \vartheta(\eta) = 0, \tag{17}
\]

\[
\theta''(\eta) + 3Prf(\eta)\vartheta(\eta) = 0,
\]

where \( Pr = \frac{\nu}{\alpha} \). The transformed boundary conditions are

\[
\eta = 0 : f(0) = 0, \quad f'(0) = 0, \quad \vartheta(0) = 1, \quad \eta \to \infty : f' \to 0, \quad \vartheta \to 0. \tag{18}
\]
To construct a simple and accurate HWCM for the problem (17) and (18), the wavelet approximations for the highest derivatives of $f$ and $\theta$ are given by

$$ f''(\eta) = \sum_{i=1}^{2M} a_i h_i(\eta). $$

$$ \theta''(\eta) = \sum_{i=1}^{2M} b_i h_i(\eta). $$

The values of $f''(\eta), f'(\eta), \theta'(\eta), f(\eta)$ and $\theta(\eta)$ can be obtained by integrating Eqs. (19) and (20) and are given by

$$ f''(\eta) = \sum_{i=1}^{2M} a_i \left( p_{i,1}(\eta) - \frac{1}{L} C_{i,1} \right). $$

$$ f'(\eta) = \sum_{i=1}^{2M} a_i \left( p_{i,2}(\eta) - \frac{1}{L} \eta C_{i,1} \right). $$

$$ f(\eta) = \sum_{i=1}^{2M} a_i \left( p_{i,3}(\eta) - \frac{1}{L} \eta^2 C_{i,1} \right). $$

$$ \theta''(\eta) = -\frac{1}{L} + \sum_{i=1}^{2M} b_i \left( p_{i,1}(\eta) - \frac{1}{L} C_{i,1} \right). $$

$$ \theta'(\eta) = -\frac{1}{L} + \sum_{i=1}^{2M} b_i \left( p_{i,2}(\eta) - \frac{1}{L} \eta C_{i,1} \right). $$

$$ \theta(\eta) = 1 - \frac{1}{L} + \sum_{i=1}^{2M} b_i \left( p_{i,2}(\eta) - \frac{1}{L} \eta C_{i,1} \right). $$

where $L$ be a sufficiently large integer. The Eqs. (19)–(25) are used in Eqs. (17) and (18) to obtain numerical solution for the ODEs. The details of the numerical procedure are shown in Section 7.

### 3.2. Laminar film condensation (LFC)

Since the fundamental contribution of Nusselt [31] concerning the laminar film condensation, many researchers have shown interest in this area in the past years. The basic Nusselt analysis ignores inertial effects in the condensed film and sub-cooling effects [32]. In this study a laminar film condensation of a saturated stream is considered on an isothermal vertical flat plate where $x$ and $y$ are the measures of the distances in the downward direction parallel and perpendicular to the plate respectively. The leading edge of the plate is located at $x = y = 0$. The coordinate system and the velocity components are shown in Fig. 2 (left). The governing equations of mass, momentum, and energy in the liquid phase are given by (see [32] for details.)

$$ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. $$

$$ \frac{\partial u}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial y} = -\nu \frac{\partial^2 u}{\partial y^2} + g \left( \frac{\partial - \rho\nu}{\rho} \right). $$

$$ \frac{\partial T}{\partial x} + \frac{1}{\rho C_v} \frac{\partial p}{\partial y} = \frac{\partial^2 T}{\partial y^2}. $$

The assumptions that the change of pressure across the film is negligible and that the velocity gradient in the cross-film direction

![Fig. 5](image1.jpg) Ex.1 (NCBLF), HWCM $\theta$ on the left and on the right $Pr = 20,000$ and on the right $Pr = 2000$, $f = 3$.

![Fig. 6](image2.jpg) Ex.1 (NCBLF), HWCM versus RKM $f$ on the left and on the right $\theta$, $J = 4$, $L = 7$. 

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Asymptotic solutions of the problems related to LFC are discussed in the recent references [2,42,1,33,43,44,4,16]. The reason of growing research in this topic is its relevance to the design of heat exchangers, heat and fluid flows for some industrial drying and cooling processes, enhanced recovery of petroleum resources, packed-bed heat exchangers, solidification, geothermal reservoirs (see Eckert [9]). Sparrow and Gregg [39] have used similarity transformation to reduce the continuity, momentum and energy equations in to a set of coupled ODEs. Some methods of solution specific to LFC are discussed in [44,32]. The similarity transformations introduced in [32] are given as,

\[ \psi = \left( \frac{u - \beta_x}{\beta} \right) g^2 \frac{x^3}{y} f(\eta), \]

\[ \theta(\eta) = \frac{1 - T_\infty}{T_w - T_\infty}, \]

\[ \eta = \frac{y}{x} \left( \frac{\beta - \beta_x}{\beta} \right) g^{1/2}. \]

In the light of these transformations, the partial differential equations (26)–(28) can be reduced to the following system of nonlinear ODEs,

\[ f'' - \frac{(f')^2}{2} + \frac{3}{4} f''' + 1 = 0, \]

\[ \theta'' + \frac{3}{2} \Pr f\theta' = 0. \]

The boundary conditions of the reduced equations are

\[ f(0) = 0, f'(0) = 0, f''(\eta) = 0, \theta(0) = 1, \theta'(0) = 0. \] (32)

The Haar wavelets approximation for the problem (31) and (32) is given as follows,

\[ f(\eta) = \sum_{i=1}^{2M} a_i \left( \frac{\eta^2}{2} \right)^i, \]

\[ \theta(\eta) = 1 - \frac{\eta}{L} + \sum_{i=1}^{2M} b_i \left( \frac{\eta}{L} \right)^i. \] (34)

The details of the numerical procedure are shown in Section 7.

3.3. Boundary layer flow and heat transfer due to a stretching sheet (BLHFFS)

A two dimensional unsteady flow and heat transfer analysis of an incompressible flow past a semi infinite stretching sheet in the region \( y > 0 \) is considered in [37,34,1]. These type of flows are of considerable importance and their applications can be found in technological processes, such as hot rolling, wire drawing, glass-fiber, paper production [1], geophysical and insulating engineering, modeling of packed sphere beds, solar power collector etc [16]. The origin is kept fixed and two equal and opposite forces are suddenly applied along \( x \)-axis. The sheet is stretched as a result of these forces which ultimately generates flow. The wall temperature \( T_w(x,t) \) of the sheet is suddenly raised from \( T_w \) to \( T_w \) which generates a sudden heat flux at the wall. The graphical view of the model is shown in Fig. 2 (right). The governing equations of mass,
The similarity transformations used in [37], are

\[ \frac{\partial u}{\partial x} + \frac{\partial w}{\partial y} = 0, \]  
\[ \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + w \frac{\partial T}{\partial y} = \frac{\partial^2 T}{\partial y^2}, \]  
\[ \frac{\partial^2 T}{\partial y^2} = \frac{\partial^2 u}{\partial y^2}. \]  

The above assumptions lead to the following initial and boundary conditions;

\[ t < 0 : u = v = 0, \quad T = T_w, \quad \text{for any } x, y; \]
\[ t \geq 0 : u = u_w(t, x), \quad v = 0, \quad T = T_w(t, x), \quad \text{or} \quad \frac{\partial T}{\partial y} = \frac{q_w(t, x)}{k}, \quad u \to 0, \quad T \to T_w \text{ as } y \to \infty. \]  

The velocity, temperature and heat flux of the sheet are defined as

\[ u_w(t, x) = \frac{c x}{1 - \gamma t}, \quad T_w(t, x) = T_w + \frac{c}{2 \nu x^2 (1 - \gamma t)^2}, \]
\[ q_w(t, x) = \frac{q_w}{2 \nu x^2 (1 - \gamma t)^2}. \]  

The similarity transformations used in [37], are

\[ \eta = \sqrt{\frac{c}{\nu (1 - \gamma t)}}, \quad \psi = \sqrt{\frac{c}{1 - \gamma t}} \psi(\eta), \]
\[ T = T_w + \frac{c}{2 \nu x^2 (1 - \gamma t)^2} \theta(\eta), \quad \text{or} \quad T = T_w + \frac{c q_w}{2 \nu x^2 (1 - \gamma t)^2} \theta(\eta). \]  

Using these transformations the governing equations of mass, momentum and energy (35)–(37) can be reduced to the following nonlinear ordinary differential equations,

\[ f'' + f f'' - (f')^2 - A(f' + \frac{1}{2} y f'') = 0, \]
\[ \theta' + Pr(f' + 2f' \theta - \frac{1}{2}(3\theta + \eta \theta')) = 0. \]  

The boundary conditions given in (39) can be written in reduced form as:

\[ f(0) = 0, \quad f'(0) = 1, \quad \theta(0) = 1, \quad \text{or} \quad \theta'(0) = 1, \]
\[ f' \to 0, \quad \theta \to 0, \quad \text{as } \eta \to \infty. \]  

The Haar wavelet approximation for the problem (41) and (42) is given as follows;

\[ f(\eta) = \eta - \eta^2 + \sum_{i=1}^{2M} \left( p_i(\eta) - \frac{1}{L} \theta^2 C_{i1} \right), \]
\[ \theta(\eta) = 1 - \frac{1}{L} \eta + \sum_{i=1}^{2M} b_i \left( p_i(\eta) - \frac{1}{L} C_{i1} \right), \]  

where \( L \) is the value of \( \eta \) at the outer edge.
4. Convergence Analysis of the Haar Wavelets

Lemma 1. Assume that \( u(x) \in L_2(\mathbb{R}) \) with the bounded first derivative on (0,1), then the error norm at \( J \)th level satisfies the following inequality

\[
\|e_J(x)\| \leq \sqrt{\frac{K}{T}} C 2^{-3(2^J-1)},
\]

where \( K, C \) are some real constants and \( M \) is a positive number related to \( J \)th resolution of the wavelet defined earlier.

Proof. The error at \( J \)th level may be defined as

\[
|e_J(x)| = |u(x) - u_J(x)| = \left| \sum_{i=1}^{2^J+1} a_i h_i(x) \right|,
\]

where \( u_J(x) = \sum_{i=1}^{2^J+1} a_i h_i(x) \).

\[
\|e_J(x)\|^2 = \sum_{i=1}^{2^J+1} |a_i|^2.
\]

But \( |a_i| \leq C 2^{-3} \max |u(\eta)| \) where \( C = \int_{\eta}^{1} |x h_2(x)| \) and \( \eta \in (k/2^J, (k+1)/2^J) \) [20,40,15].

\[
\|e_J(x)\|^2 \leq \sum_{i=1}^{2^J+1} KC 2^{-3(i)},
\]

where \( |u(\eta)| \leq K \) \( \forall \eta \in (0,1) \).

\[
\|e_J(x)\|^2 \leq KC 2^{-3(2^J-1)} = \sqrt{\frac{K}{T}} C 2^{-3(2^J-1)}. \tag{48}
\]

From the Eq. (48), it is obvious that the error bound is inversely proportional to the level of resolution of Haar wavelet. This ensures the convergence of Haar wavelet approximation when \( J \) is increased.

5. The algorithm

The aim of this section is to implement the Haar wavelet algorithm for a coupled system of nonlinear ODEs with different sets of boundary conditions. The procedure to find the approximate solution of the nonlinear boundary-value problem.
Step 4 For $j = 1, 2, ..., 2M$, the Case(i)–Case(iii) solutions are obtained by solving the system (17), (18), (31), (32), (41) and (42) for $4M$ unknown wavelets coefficients $a_i$ and $b_i$ at the collocation points $j$, $j = 1, 2, ..., 2M$.

6. Viscoelastic fluid with known exact solution

In order to assess the accuracy of the method in terms of root mean square square error norm and infinity norm, we consider a viscoelastic fluid [30] with the known analytical solution.

$$\frac{d^2 F}{d\eta^2} - \left(\lambda_0 \beta^2 + \beta\right) F = 0, \quad F(0) = U_0, \quad F(\infty) = 0.$$  \hspace{1cm} (49)

The analytical solution of the problem (49) is given by

$$F(\eta) = U_0 \exp \left(-\sqrt{-\frac{\lambda_0 \beta^2 + \beta}{\eta}}\right).$$  \hspace{1cm} (50)

The Haar solution for the problem (49) is obtained as a procedure adopted for other cases and is given as follows,

$$F = U_0 - \frac{U_0}{L} \eta + \sum_{i=1}^{2M} a_i \left(\eta - \frac{\eta}{L} C_{i,1}\right).$$  \hspace{1cm} (51)

In our calculations we have taken $\nu = U_0 = \lambda_0 = \beta = 1$.

7. Numerical results and discussion

In this section we present numerical and graphical results obtained from HWCM applied to the three different cases (i)–(iii) (NCBLF, LFC and BLFHTSS) along with the study of visco-elastic fluid model with the known exact solution. The algorithm is implemented in Mathematica 7.0 software where as the ODE solver NDsolve is used in the case of RKM. The CPU time is calculated by

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Ex. 4, Error Norms for $F$ on the interval [0,7] versus different levels of resolution $J$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level of resolution</td>
<td>$L_\infty$</td>
</tr>
<tr>
<td>$J = 2$</td>
<td>$8.17 \times 10^{-3}$</td>
</tr>
<tr>
<td>$J = 3$</td>
<td>$2.26 \times 10^{-3}$</td>
</tr>
<tr>
<td>$J = 4$</td>
<td>$7.15 \times 10^{-4}$</td>
</tr>
<tr>
<td>$J = 5$</td>
<td>$1.82 \times 10^{-4}$</td>
</tr>
<tr>
<td>$J = 6$</td>
<td>$4.83 \times 10^{-5}$</td>
</tr>
<tr>
<td>$J = 7$</td>
<td>$1.23 \times 10^{-6}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 2</th>
<th>Ex. 4, Error Norms for $F$ of RKM for different values of $L$ and $dt = 0.001, 0.0001$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length of the computational domain</td>
<td>$L_\infty$</td>
</tr>
<tr>
<td>$L = 1$</td>
<td>$2.43 \times 10^{-1}$</td>
</tr>
<tr>
<td>$L = 2$</td>
<td>$5.91 \times 10^{-2}$</td>
</tr>
<tr>
<td>$L = 5$</td>
<td>$8.49 \times 10^{-4}$</td>
</tr>
<tr>
<td>$L = 7$</td>
<td>$5.05 \times 10^{-5}$</td>
</tr>
<tr>
<td>$L = 30$</td>
<td>Unstable</td>
</tr>
<tr>
<td>$L = 30, J = 4$ (HWCM)</td>
<td>$1.85 \times 10^{-1}$</td>
</tr>
</tbody>
</table>

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employing Timing command. Intel Core 2 Duo processor with 4 GB RAM is used for the computation.

The natural convection boundary layer flow problem NCBLF discussed in the previous section was solved numerically by [30] using FDM with the small value of $Pr = 0.72$ only. Similar cases have recently been solved by HAM [11] and implicit finite difference [14] for small values of $Pr$. The solution produced through HWCM for the natural convection problem NCBLF is shown in Figs. 3–8 for larger values of $Pr$. In Figs. 3–5 a comparison of HWCM versus RKM is shown. From these figures it is clear that HWCM produces stable results for larger values of $Pr$ where as RKM produces unstable results for these cases. The initial interval step size for RKM is chosen as $dt = 0.001$ and $dt = 0.0001$. In Fig. 6 comparison of HWCM versus RKM is shown for $L = 7$. It is clear from the figure that RKM produces unstable results in this case as well. The numerical solution of initial slopes generated through HWCM is given in Table 5 alongside FDM solution [30]. Comparison of the HWCM with RKM for small values of $Pr$ is given in Table 6. It is clear from these tables that the Haar wavelets based algorithm agrees well with finite difference solution [30] and RKM for small values of $Pr$. The results obtained from [30,11,14] are restricted only to small values of $Pr$ where as the present method can handle larger values of the parameter. From Fig. 7 (left) it is clear that with the increase in $Pr$, the thickness of the layer decreases. Fig. 7 (right) represents the variations of the velocity distribution in the boundary layer profiles for various values of $Pr$. It is clear from this figure that the boundary layer flow attains minimum velocity for higher values of $Pr$. Also from Fig. 8 (left) it is obvious that the thermal boundary layer decreases as well with the increase in the values of $Pr$. Fig. 8 (right) shows that the rate of change of the thermal boundary layer is prominent in the middle. This is in conformity with the fact that the thermal conductivity of the film decreases with increase in the value of $Pr$ keeping in view the definition of $Pr$. Consequently, the heat transfer through the flow decreases with the increase in the value of $Pr$.

The numerical results corresponding to the laminar film condensation and free convection boundary layer flow (LFC) are shown in Figs. 9–11. This problem has recently been solved by HAM [44]. In Fig. 9 (left) there is no effect of changing values of $Pr$ on $f$. In Fig. 9 (right) derivative of $f$ for $Pr = 100$ is shown. From Fig. 10 (left) it is clear that with increase of $Pr$ the thickness of the thermal boundary layer decreases. Fig. 10 (right) it is clear that the variable $\theta$ changes rapidly in the middle as $Pr$ changes values from 1 to 100. Like the previous case, HWCM based algorithm produces stable numerical solution for fluid flows with larger values of $Pr$, which is quite challenging for asymptotic methods like HAM, HPM and DTM and numerical method RKM. These methods often diverge for high values of $Pr$. Even well established RKM produces unstable solution for large values of $Pr$. Excellent agreement of HWCM and RKM is found for both $f$ and $\theta$ for small values of $Pr$ which is shown in Table 7. In Fig. 11 results of HWCM versus RKM are shown for high values of $Pr = 1500$, 2000. It is clear from these figures that the instability appears for $\theta$-component of the solution in the case of RKM. A set of different step sizes ranging from $10^{-2}$ to $10^{-5}$ is used

Table 3
<table>
<thead>
<tr>
<th>$Pr$</th>
<th>HWCM time in Secs.</th>
<th>RKM time in Secs.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.321</td>
<td>0.21</td>
</tr>
<tr>
<td>10</td>
<td>0.124</td>
<td>0.37</td>
</tr>
<tr>
<td>100</td>
<td>0.141</td>
<td>0.89</td>
</tr>
<tr>
<td>150</td>
<td>0.141</td>
<td>1.25</td>
</tr>
<tr>
<td>170</td>
<td>0.141</td>
<td>1.36</td>
</tr>
<tr>
<td>300</td>
<td>0.141</td>
<td>1.59</td>
</tr>
<tr>
<td>400</td>
<td>0.141</td>
<td>2.00</td>
</tr>
<tr>
<td>600</td>
<td>0.141</td>
<td>2.30</td>
</tr>
<tr>
<td>800</td>
<td>0.141</td>
<td>14.21 (Solution diverges)</td>
</tr>
<tr>
<td>1000</td>
<td>0.141</td>
<td>21.8 (Solution diverges)</td>
</tr>
<tr>
<td>1500</td>
<td>0.141</td>
<td>24.6 (Solution diverges)</td>
</tr>
</tbody>
</table>

Table 4
<p>| Ex. 2, CPU time, HWCM versus RKM, $2M = 8$, $L = 1$. |</p>
<table>
<thead>
<tr>
<th>$Pr$</th>
<th>HWCM time in Secs.</th>
<th>RKM time in Secs.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.29</td>
<td>0.2</td>
</tr>
<tr>
<td>10</td>
<td>0.11</td>
<td>0.4</td>
</tr>
<tr>
<td>50</td>
<td>0.11</td>
<td>0.6</td>
</tr>
<tr>
<td>100</td>
<td>0.11</td>
<td>2.4</td>
</tr>
<tr>
<td>200</td>
<td>0.11</td>
<td>6.4</td>
</tr>
<tr>
<td>500</td>
<td>0.11</td>
<td>26.6</td>
</tr>
<tr>
<td>1000</td>
<td>0.11</td>
<td>Solution diverges</td>
</tr>
</tbody>
</table>

Table 5
<p>| Numerical solution of Natural Convection Boundary Layer Flow (NCBLF) for $Pr = 0.72$. |</p>
<table>
<thead>
<tr>
<th>Present Method</th>
<th>$2M$</th>
<th>$\frac{d^2u(0)}{dx^2}$</th>
<th>$\frac{d\theta(0)}{dx}$</th>
<th>Iter</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>8</td>
<td>0.6703</td>
<td>0.5057</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>16</td>
<td>0.6732</td>
<td>0.5050</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>32</td>
<td>0.6739</td>
<td>0.5048</td>
<td>4</td>
</tr>
</tbody>
</table>

Table 6
<p>| Ex. 1, Comparison of HWCM and RKM for $f$, $J = 3$, $L = 1$, $Pr = 3$. |</p>
<table>
<thead>
<tr>
<th>$\eta$</th>
<th>$f$</th>
<th>RKM</th>
<th>HWCM</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>HWCM</td>
<td>RKM</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0.1</td>
<td>0.00145326</td>
<td>0.00147658</td>
<td>0.894057</td>
</tr>
<tr>
<td>0.2</td>
<td>0.00519972</td>
<td>0.00529088</td>
<td>0.788271</td>
</tr>
<tr>
<td>0.3</td>
<td>0.0103994</td>
<td>0.0105972</td>
<td>0.682997</td>
</tr>
<tr>
<td>0.4</td>
<td>0.0163197</td>
<td>0.0166537</td>
<td>0.578705</td>
</tr>
<tr>
<td>0.5</td>
<td>0.0223356</td>
<td>0.0228227</td>
<td>0.475936</td>
</tr>
<tr>
<td>0.6</td>
<td>0.0279277</td>
<td>0.0285706</td>
<td>0.375206</td>
</tr>
<tr>
<td>0.7</td>
<td>0.0326819</td>
<td>0.0334678</td>
<td>0.276967</td>
</tr>
<tr>
<td>0.8</td>
<td>0.0362896</td>
<td>0.0371863</td>
<td>0.181565</td>
</tr>
<tr>
<td>0.9</td>
<td>0.0385254</td>
<td>0.0395007</td>
<td>0.0892148</td>
</tr>
<tr>
<td>1.0</td>
<td>0.0392829</td>
<td>0.040284</td>
<td>0.0000000</td>
</tr>
</tbody>
</table>

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in the case of RKM. The results are unstable irrespective of the magnitude of the step size. Contrary to the RKM, HWCM produces stable results for each case irrespective of the magnitude of Pr.

Numerical results related to the two dimensional boundary layer flow and heat transfer due to a stretching sheet BLFHTSS are shown in Figs. 12 and 13. This problem has been solved by asymptotic methods [37,34,1]. It is obvious from Fig. 12 (left) that variation of the parameter $A$ has a little effect on $f$ whereas the thermal boundary layer (right) decreases considerably as the value of the parameter $A$ is increased while keeping Pr fixed. The constant $A = \frac{X}{t}$ is a non-dimensional parameter which measures the flow and the heat transfer unsteadiness. To assess the accuracy of the HWCM, a comparison with standard RKM based ODE solver in Mathematica 7 is carried out and the results are shown in Table 8. Excellent agreement between HWCM and RKM is found for both $f$ and $\theta$ for small values of Pr. Like the previous two cases HWCM is more efficient than RKM for large values Pr. To be specific in the case of NCBLF, HWCM 174 time more faster than RKM for Pr = 1500 and 16 times more faster for Pr = 600. Similarly in the case of LFC, HWCM is 241 times more faster than RKM for Pr = 500 and 58 times more faster for Pr = 200. The main reason for the computational efficiency of HWCM is that the main components of the method given in Eqs. (1)–(11) are evaluated once and stored in the memory for subsequent computations.

8. Conclusion

In this paper, a simple and straightforward numerical technique based on Haar wavelets collocation is proposed for the numerical solution of a variety of systems of boundary value problems arising in natural convection boundary layer flows. It has been found that the developed HWCM provides a superior performance in comparison with classical RKM and asymptotic method like HAM, for high values of Pr. Based on the findings presented in the previous sections, we summarize the outcomes of this study as follow:

**Table 7**

<table>
<thead>
<tr>
<th>$\eta$</th>
<th>$f$</th>
<th>$\theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>HWCM</td>
<td>RKM</td>
<td>HWCM</td>
</tr>
<tr>
<td>0</td>
<td>0.00000000</td>
<td>0.00000000</td>
</tr>
<tr>
<td>0.1</td>
<td>0.00664457</td>
<td>0.00064449</td>
</tr>
<tr>
<td>0.2</td>
<td>0.01791220</td>
<td>0.01791140</td>
</tr>
<tr>
<td>0.3</td>
<td>0.03880750</td>
<td>0.03880280</td>
</tr>
<tr>
<td>0.4</td>
<td>0.06633280</td>
<td>0.06632970</td>
</tr>
<tr>
<td>0.5</td>
<td>0.09950780</td>
<td>0.09950280</td>
</tr>
<tr>
<td>0.6</td>
<td>0.13735400</td>
<td>0.13734700</td>
</tr>
<tr>
<td>0.7</td>
<td>0.17890800</td>
<td>0.17889700</td>
</tr>
<tr>
<td>0.8</td>
<td>0.22321600</td>
<td>0.22320600</td>
</tr>
<tr>
<td>0.9</td>
<td>0.26934500</td>
<td>0.26926200</td>
</tr>
<tr>
<td>1.0</td>
<td>0.31637400</td>
<td>0.31635100</td>
</tr>
</tbody>
</table>

**Table 8**

<table>
<thead>
<tr>
<th>$\eta$</th>
<th>$f$</th>
<th>$\theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>HWCM</td>
<td>RKM</td>
<td>HWCM</td>
</tr>
<tr>
<td>0</td>
<td>0.00000000</td>
<td>0.00000000</td>
</tr>
<tr>
<td>0.1</td>
<td>0.0826354</td>
<td>0.0826274</td>
</tr>
<tr>
<td>0.2</td>
<td>0.139788</td>
<td>0.139766</td>
</tr>
<tr>
<td>0.3</td>
<td>0.180772</td>
<td>0.180737</td>
</tr>
<tr>
<td>0.4</td>
<td>0.211096</td>
<td>0.211051</td>
</tr>
<tr>
<td>0.5</td>
<td>0.231399</td>
<td>0.234086</td>
</tr>
<tr>
<td>0.6</td>
<td>0.252024</td>
<td>0.251968</td>
</tr>
<tr>
<td>0.7</td>
<td>0.266082</td>
<td>0.266028</td>
</tr>
<tr>
<td>0.8</td>
<td>0.277049</td>
<td>0.277003</td>
</tr>
<tr>
<td>0.9</td>
<td>0.284963</td>
<td>0.284934</td>
</tr>
<tr>
<td>1.0</td>
<td>0.288429</td>
<td>0.288416</td>
</tr>
</tbody>
</table>

Numerical accuracy of the method is measured in terms $L_e$ and $L_{rms}$ error norms versus increasing level of resolution of the Haar wavelets and is shown in Tables 1 and 2. The accuracy of the method increases considerably by increasing the level of resolution $J$. Numerical convergence of the algorithm in terms $L_{rms}$ and different levels of wavelet resolution is shown in Fig. 14. Better performance in terms of accuracy and rapid convergence is archived through HWCM.

In Tables 3 and 4 and Fig. 15, CPU times of HWCM versus RKM for the cases NCBLF and LFC are calculated. It is clear from these tables that HWCM is more efficient than RKM for large values Pr. To be specific in the case of NCBLF, HWCM 174 time more faster than RKM for Pr = 1500 and 16 times more faster for Pr = 600. Similarly in the case of LFC, HWCM is 241 times more faster than RKM for Pr = 500 and 58 times more faster for Pr = 200. The main reason for the computational efficiency of HWCM is that the main components of the method given in Eqs. (1)–(11) are evaluated once and stored in the memory for subsequent computations.

Fig. 15. Ex.1 (NCBLF), CPU time in Sec. HWCM versus RKM (Left), Ex.2 (LFC), CPU time in Sec. HWCM versus RKM (Right) for different values of Pr.
Better accuracy of HWCM can be obtained by increasing the
gramme Group Modeling of Materials and Processes). The
the Slovenian Grant Agency through grant no. P2-0379 (Pro-

Contrary to RKM, HWCM does not require conversion of

Unlike RKM, the boundary value problem need not to be
reduced into a system of first order ODEs in the case of HWCM,

Variety of boundary conditions can be handled with equal ease,

Better accuracy of HWCM can be obtained by increasing the resolution J of the waves,

The disadvantage of Haar wavelets is that the first and higher-
order derivatives are not continuous and hence the highest order derivative is approximated by Haar functions using integral approach instead of direct differentiation, to recover approximation for the original function.

A simple applicability and a fast convergence of the Haar wavelets provide a solid foundation for using these functions in the context of numerical approximation of integral equations, partial differential equations and ordinary differential equations.

The authors are grateful to the financial support provided from the Slovenian Grant Agency through grant no. P2-0379 (Programme Group Modeling of Materials and Processes). The first author is also thankful to NWFP UET Peshawar Pakistan for sabbatical leave. We are also thankful to the reviewers for their valuable suggestions.

(i) HWCM provides accurate solution in the case of flows with high Prandtl numbers, Reynolds numbers, Hartmann numbers and Rayleigh numbers,

(ii) HWCM is more suitable for the numerical solution of boundary value problems defined on long intervals. These type of boundary value problems have a special significance in the context of natural convection boundary layer fluid flows,

(iii) HWCM needs less CPU time as the major blocks of HWCM are calculated only once and used in the subsequent computations repeatedly,

(iv) Contrary to RKM, HWCM does not require conversion of a boundary value problem into initial value problem by using shooting like procedure, and hence the boundary-value problem is not integrated as an initial value problem with guesses for the unknown initial values. This results in a more stable behavior of HWCM. The reasons for stability are that it does not use shooting technique. In the case of RKM the shooting procedure causes unstable results for the problems with the large computational domain,

(v) Unlike RKM, the boundary value problem need not to be
reduced into a system of first order ODEs in the case of HWCM,

(vi) Variety of boundary conditions can be handled with equal ease,

(vii) HWCM provides accurate solution in the case of

References

Acknowledgments

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