Towards a mesh-free computation of transport phenomena

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Abstract

This paper formulates a computational procedure based on the Trefftz method for the solution of nonlinear transport phenomena. This new unified approach is particularly important when solving coupled, nonlinear, inhomogenous, anisotropic, multiphase, and multifield heat and mass transfer problems. Physical system represents the general transport equation, standing for a broad spectra of mass, energy, momentum, and species transfer problems. This equation is cast into non-linear Poisson form and expanded with respect to the transport variable. Fully implicit time-discretization is used. The particular solution method is applied as a general solution framework. The solution of the inhomogenous part is based on the radial basis function global approximation, and the solution of the homogenous part is based on the Trefftz method Laplace equation fundamental solution global approximation. The discrete approximate method results in a global point-collocation based grid and hence eliminates the need for polygonization of the computational boundary and domain. © 2002 Elsevier Science Ltd. All rights reserved.

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1. Introduction

1.1. Impulses

The development of efficient as well as simple algorithms for the numerical solution of partial differential equations (PDEs) is of major interest in applied sciences and engineering. The most popular discrete approximate methods for PDEs are nowadays the finite difference (FDM), the finite volume (FVM), the finite element (FEM), the spectral (SM), and the boundary element (BEM) methods. Despite the powerful features of these methods, there are often substantial difficulties in applying them to realistic, geometrically complex three-dimensional transient problems [1]. A common drawback of the mentioned methods is the need to create a polygonization, either in the domain and/or on its boundary. This type of meshing is often the most time consuming part of the solution process and is far from being fully automated.

1.2. Polygonization in multiphase transport phenomena problems

The numerical solution of coupled heat and mass transfer problems is becoming increasingly important as a result of the computational modelling needs in diverse modern technologies. A broad class [2] of such heat, mass, momentum, and solute transfer problems involves two or more phases, separated by free (steady state) or moving (transient) interphase boundaries. Due to the existence of complex shaped interphase boundaries, most of the numerical simulations of engineering gas–liquid and liquid–solid two-phase flows conducted so far have been based on averaged field equations with constitutive interphase relations solved on a fixed mesh. However, the diversity of the possible involved length scales, inhomogeneities, and anisotropies, usually requires the adaptation of the mesh with respect to high field gradients and subsequent re-meshing. Recent rapid progress in computer performance however gives us a large prospect of realizing more detailed numerical simulation of multiphase systems. Physically sound information can properly be perceived only from the numerical approaches which explicitly take into account the moving boundaries. The principal bottleneck in these type of numerical methods is the time consuming re-meshing of the evolving interphase boundaries and phase domains which limits such methods to problems with quite trivial phase patterns. The polygonization problem is thus even more pronounced in such type of front-tracking approach.
1.3. Fundamental solution based methods

One method for alleviating the described difficulties is to use the BEM [3], which requires only boundary polygonization in special cases like the potential flow. Realistic viscous laminar or turbulent situations may not be solved by these methods without domain polygonization either and lead to the boundary-domain integral methods (BDIM). See Fig. 1a for illustration of typical BDIM mesh schematics. In recent years there has been considerable interest to develop ‘mesh-free’ methods which do not involve extensive polygonization. Most common among such methods, derived from BDIM, is the dual reciprocity boundary element method (DRBEM) [4]. See Fig. 1b for illustration of typical DRBEM mesh schematics. In this method, the weighted residual formulation of the governing PDE is reduced to a set of boundary integrals only by using global interpolation of the domain integrals. One of the drawbacks of boundary integral methods is that formulation involves the evaluation of hypersingular, strongly singular, singular or near-singular integrals. The accurate evaluation of these integrals is usually computationally difficult and expensive. To circumvent the problem of having to evaluate boundary integrals, a class of boundary collocation methods classified under the generic name ‘Trefftz method’ can be used. Here the solution is represented using layer potentials on non-physical surfaces, thereby by-passing the need to evaluate any integrals. By adopting such a method, one also avoids the problem of complicated surface polygonization, required in both the traditional BDIM as well in the DRBEM [5]. This paper represents a logical upgrading of our DRBEM solution [6] of the general transport equation in sense of solving it on a complete mesh-free basis. The simple grid schematics of the present meshless method is illustrated in Fig. 1c. The developments of Golberg [7] for linear and Balakrishnan and Ramachandran [8] for non-linear Poisson equation have been used as a starting point for this study. Present paper explicitly takes into account all engineering boundary condition types and specifics of the Poisson equation, originating from the general transport equation.

2. Governing equations

For the present purposes, a transport phenomena problem can be briefly described in a general manner as the numerical solution of Eulerian transport equation, defined on a fixed domain $\Omega$ with boundary $\Gamma$, of the kind

$$\frac{\partial}{\partial t} [\varrho \mathcal{F}(\Phi)] + \nabla \cdot [\varrho \mathbf{v} \mathcal{F}(\Phi)] = -\nabla \cdot (-\mathbf{D} \nabla \Phi) + S$$

with $\varrho$, $\Phi$, $t$, $\mathbf{v}$, $S$ and $\mathbf{D}$ standing for density, transport variable, time, velocity, source, and diffusion matrix

$$\mathbf{D} = \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{pmatrix}.$$ 

The transport variable stands, for instance for the velocity
component in each co-ordinate direction, or temperature, or the mass fraction of a chemical species. The function \( \mathcal{F} \) denotes the relation between the transported and the diffused variable such as for example relation between the enthalpy and the temperature.

The solution of the governing equation for the dependent variable \( \Phi \) at final time \( t = t_0 + \Delta t \) is sought, where \( t_0 \) represents the initial time and \( \Delta t \) the positive time increment. The solution is constructed by the initial and boundary conditions that follow. The initial value of transport variable \( \Phi(p, t_0) \) at a point with position vector \( p \) and time \( t_0 \) is defined through the known function \( \Phi_0 \):

\[
\Phi(p, t) = \Phi_0; \quad p \in \Gamma \cup \Omega. \tag{3}
\]

The boundary \( \Gamma \) is divided into not necessarily connected parts \( \Gamma^\varnothing, \Gamma^\psi \) and \( \Gamma^\varphi \)

\[
\Gamma = \Gamma^\varnothing \cup \Gamma^\psi \cup \Gamma^\varphi, \tag{4}
\]

with Dirichlet, Neumann and Robin type boundary conditions, respectively. These boundary conditions are at the boundary point \( p \) with normal \( \mathbf{n} \), and time \( t_0 \leq t \leq t_0 + \Delta t \) defined through known functions \( \Phi^\varnothing, \Phi^\psi, \Phi^\varphi \) and \( \Phi^\text{ref} \):

\[
\Phi = \Phi^\varnothing; \quad p \in \Gamma^\varnothing, \tag{5}
\]

\[
\frac{\partial}{\partial n_r} \Phi = \Phi^\psi; \quad p \in \Gamma^\psi,
\]

\[
\frac{\partial}{\partial n_r} \Phi = \Phi^\varphi(\Phi - \Phi^\text{ref}); p \in \Gamma^\varphi.
\]

The involved parameters of the governing equation and boundary conditions are assumed to depend on the transport variable, space and time. The solution procedure thus inherently involves iterations. The governing equation is transformed as follows. The diffusion matrix

\[
D = D_I + D', \tag{6}
\]

is split into constant isotropic part \( D_I \), with \( I \) denoting identity matrix, and the remaining nonlinear anisotropic part \( D' \):

\[
I = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

\[
D' = \begin{pmatrix}
d_{11} - D & d_{12} & d_{13} \\
d_{21} & d_{22} - D & d_{23} \\
d_{31} & d_{32} & d_{33} - D
\end{pmatrix}. \tag{7}
\]

The transport equation is subsequently cast into Poisson form

\[
\nabla^2 \Phi = \theta + \nabla \cdot \Theta, \tag{8}
\]

with

\[
\theta = \left\{ \frac{\partial}{\partial t} [\mathcal{F}(\Phi)] - \mathcal{S} \right\} / D, \quad \Theta = [\mathcal{F}(\Phi) - D \nabla \Phi] / D. \tag{9}
\]

The inhomogenous terms are Taylor expanded as

\[
\theta = \tilde{\theta} + \tilde{\theta}_\Phi (\Phi - \bar{\Phi}), \quad \Theta = \bar{\Theta} + \bar{\Theta}_\Phi (\Phi - \bar{\Phi}), \tag{10}
\]

with ‘bar’ denoting value at previous iteration. The final form of the transformed equation, suitable for iterative solution, becomes

\[
\nabla^2 \Phi = \theta + \nabla \cdot \Theta = \tilde{\theta} + \tilde{\theta}_\Phi (\Phi - \bar{\Phi}) + \nabla \cdot \bar{\Theta} + \nabla \cdot \tilde{\Theta}_\Phi (\Phi - \bar{\Phi}). \tag{11}
\]

3. Solution procedure

3.1. Time discretization

The time discretization is made (for simplicity of presentation only) in a

\[
\theta = \left[ \frac{\mathcal{F}(\Phi) - \mathcal{F}(\Phi_0)}{\Delta t} - \mathcal{S} \right] / D, \tag{12}
\]

fully implicit (backward Euler) manner where subscript 0 represents the value at the initial time.

3.2. The method of particular solutions

The solution procedure is built within the framework of the method of particular solutions where the solution \( \Phi \) is represented in terms of the particular solution \( \Phi^\varnothing \) and homogenous solution \( \Phi^\psi \)

\[
\Phi = \Phi^\varnothing + \Phi^\psi. \tag{13}
\]

The particular solution of the problem satisfies the Poisson equation

\[
\nabla^2 \Phi^\varnothing = \tilde{\theta} + \tilde{\theta}_\Phi (\Phi - \bar{\Phi}) + \nabla \cdot \bar{\Theta} + \nabla \cdot \tilde{\Theta}_\Phi (\Phi - \bar{\Phi}), \tag{14}
\]

and not necessarily the boundary conditions. The homogenous solution satisfies the Laplace equation

\[
\nabla^2 \Phi^\psi = 0, \tag{15}
\]

with the modified boundary conditions, composed from the original ones and the particular solution

\[
\Phi^\psi = \Phi^\psi; \quad p \in \Gamma^\varnothing, \tag{16}
\]

\[
\frac{\partial}{\partial n_r} \Phi^\psi = \Phi^\psi - \frac{\partial}{\partial n_r} \Phi^\varnothing; \quad p \in \Gamma^\psi,
\]

\[
\frac{\partial}{\partial n_r} \Phi^\psi = \Phi^\varphi(\Phi - \Phi^\text{ref}) - \frac{\partial}{\partial n_r} \Phi^\varphi; \quad p \in \Gamma^\varphi.
\]

The solution is established in \( N_{\text{col}} \) collocation points \( p_j \).
The first subscript index is not summed on) is used in the present text. The Einstein summation convention (any index that is repeated twice in a product is summed on, the underlined index is not summed on) is used in the present text. The subscript $i$ is running over $x$ and $y$. The indices are running as $i = 1, 2, ..., N_{12}$, $j, l = 1, 2, ..., N$, $i, k, l = 1, 2, ..., N_{12}$, $j, n = 1, 2, ..., N$ if not stated otherwise. $N$ and $N_{12}$ are defined in Sections 3.3 and 3.4, respectively.

### 3.3. Construction of the particular solution

Let us first concentrate on treatment of the particular solution. The inhomogeneous term is approximated by the $n$ global approximation functions $\psi_n(p)$ and their coefficients $\xi_n$. This term can subsequently be written in the form

$$\tilde{\delta}(p) + \tilde{\theta}(p)[\tilde{\phi}(p) - \tilde{\phi}(p)] + \nabla \tilde{\Theta}(p) + \nabla \tilde{\Theta}_\phi(p)\tilde{\phi} + \nabla \tilde{\Theta}_\phi(p)\tilde{\phi} = \psi_n(p)\xi_n.$$  \hspace{1cm} (20)

The coefficients $\xi_n$ are calculated from the system of $N$ algebraic equations

$$\Psi_{jn}\xi_n = b_j.$$ \hspace{1cm} (21)

The first $N_{12}$ equations are obtained through collocation of Eq. (20) in collocating points $p_i$, distributed over $\Gamma$ and $\Omega$

$$\Psi_{jn} = \psi_n(p_i).$$ \hspace{1cm} (22)

The remaining $N - N_{12}$ equations are obtained through constraints

$$\Psi_{jn} = \psi_n(p_i), \quad j = N_{12} + 1, N_{12} + 2, ..., N,$$ \hspace{1cm} (23)

where $b_j = 0$; $j = N_{12} + 1, N_{12} + 2, ..., N,$

where we assumed

$$\psi_n(p_i) = 0; \quad j = N_{12} + 1, N_{12} + 2, ..., N.$$ \hspace{1cm} (24)

The coefficients $\xi_n$ can be expressed through inversion of

the system (Eq. (21)) that gives

$$\xi_n = \Psi_{mn}^{-1}[\tilde{\delta}_n + \tilde{\theta}_n\tilde{\phi}_n(\tilde{\phi}_i - \tilde{\phi}_i) + \nabla \tilde{\Theta}_i + \nabla \tilde{\Theta}_\phi(\tilde{\phi}_i - \tilde{\phi}_i)],$$ \hspace{1cm} (25)

and the Eq. (20) can be written in the form

$$\tilde{\delta}(p) + \tilde{\theta}(p)[\tilde{\phi}(p) - \tilde{\phi}(p)] + \nabla \tilde{\Theta}(p) + \nabla \tilde{\Theta}_\phi(p)\tilde{\phi} + \nabla \tilde{\Theta}_\phi(p)\tilde{\phi}$$

$$= \psi_n(p)\Psi_{mn}^{-1}[\tilde{\delta}_n + \tilde{\theta}_n\tilde{\phi}_n(\tilde{\phi}_i - \tilde{\phi}_i)] + \nabla \tilde{\Theta}(p) + \nabla \tilde{\Theta}_\phi(p)[\tilde{\phi}(p) - \tilde{\phi}(p)].$$ \hspace{1cm} (26)

The expression $\nabla \tilde{\Theta}$ in Eq. (20) can be calculated as follows

$$\nabla \tilde{\Theta}_i = \frac{\partial}{\partial p_i} \tilde{\Theta}_i.$$ \hspace{1cm} (27)

By making similar global approximation function expansions like in Eq. (20) one can write

$$\tilde{\Theta}_i(p) = \psi_n(p)\xi_n,$$ \hspace{1cm} (28)

with the coefficients $\xi_n$ calculated equivalently as in Eq. (25)

$$\xi_n = \Psi_{mn}^{-1}\tilde{\Theta}_i.$$ \hspace{1cm} (29)

The global interolation of the $\xi$ component of $\Theta$ thus reads

$$\tilde{\Theta}_i(p) = \psi_n(p)\Psi_{mn}^{-1}\tilde{\Theta}_i,$$ \hspace{1cm} (30)

and the global interolation of the $\xi$ component of $\nabla \tilde{\Theta}$ is

$$\nabla \tilde{\Theta}_i = \frac{\partial}{\partial p_i} \psi_n(p)\Psi_{mn}^{-1}\tilde{\Theta}_i.$$ \hspace{1cm} (31)

The final divergence expressions in collocation points $p_i$ are

$$\nabla \tilde{\Theta}_i = \frac{\partial}{\partial p_i} \psi_n(p)\Psi_{mn}^{-1}\tilde{\Theta}_i,$$

$$\nabla \tilde{\Theta}_\phi = \frac{\partial}{\partial p_i} \psi_n(p)\Psi_{mn}^{-1}\tilde{\Theta}_\phi.$$ \hspace{1cm} (32)

The particular solution Poisson problem can be represented in the form

$$\nabla^2 \tilde{\phi}_n(p) = \psi_n(p)\Psi_{mn}^{-1}[\tilde{\delta}_n + \nabla \tilde{\Theta}_i + (\tilde{\theta}_n + \nabla \tilde{\Theta}_\phi)[\tilde{\phi}_i - \tilde{\phi}_i]].$$ \hspace{1cm} (33)

It is solved by defining the harmonic functions $\hat{\psi}_n(p)$

$$\nabla^2 \hat{\psi}_n(p) = \psi_n(p),$$ \hspace{1cm} (34)

which allow the particular solution to be explicitly extracted

$$\tilde{\phi}_n(p) = \hat{\psi}_n(p)\Psi_{mn}^{-1}[\tilde{\delta}_n + \nabla \tilde{\Theta}_i + (\tilde{\theta}_n + \nabla \tilde{\Theta}_\phi)[\tilde{\phi}_i - \tilde{\phi}_i]].$$ \hspace{1cm} (35)
The following vector and matrix are introduced
\[
\mathbf{v}_n^{-1} = \mathbf{v}_m^{-1} [\mathbf{\bar{D}} + \nabla \cdot \mathbf{\bar{G}} - (\mathbf{\bar{D}} \cdot \phi_i + \nabla \cdot \mathbf{\bar{G}} \phi_i) \mathbf{\bar{D}}],
\]
\[
\mathbf{\bar{v}}_n^{-1} = \mathbf{v}_m^{-1} (\mathbf{\bar{D}} \cdot \phi_i + \nabla \cdot \mathbf{\bar{G}} \phi_i),
\]
in order to compress the notation of the particular solution (Eq. (35))
\[
\mathbf{d}_n = \mathbf{d}_m + \mathbf{\bar{v}}_n^{-1} \mathbf{\bar{v}}_m^{-1} \mathbf{\bar{D}}' \mathbf{\bar{D}}/C^2_22
\]
\[
\mathbf{v}_m^{-1} (\mathbf{\bar{D}}' \mathbf{\bar{D}}/C^2_22).
\]

3.4. Construction of the homogenous solution

The homogenous solution satisfies the Laplace equation with the modified boundary conditions, based on the boundary conditions of the original problem and the particular solution. The discretized modified boundary conditions (Eq. (16)) with the discretized particular solution (Eq. (39)) become
\[
\mathbf{d}_n = \mathbf{d}_m + \mathbf{\bar{v}}_n^{-1} \mathbf{\bar{v}}_m^{-1} \mathbf{\bar{D}}' \mathbf{\bar{D}}/C^2_22
\]
\[
\mathbf{v}_m^{-1} (\mathbf{\bar{D}}' \mathbf{\bar{D}}/C^2_22).
\]

The Dirichlet and Robin boundary conditions for the particular solution can be reformulated into
\[
\mathbf{d}_n = \mathbf{d}_m + \mathbf{\bar{v}}_n^{-1} \mathbf{\bar{v}}_m^{-1} \mathbf{\bar{D}}' \mathbf{\bar{D}}/C^2_22
\]
\[
\mathbf{v}_m^{-1} (\mathbf{\bar{D}}' \mathbf{\bar{D}}/C^2_22).
\]

by defining
\[
\mathbf{d}_n = \mathbf{d}_m + \mathbf{\bar{v}}_n^{-1} \mathbf{\bar{v}}_m^{-1} \mathbf{\bar{D}}' \mathbf{\bar{D}}/C^2_22
\]
\[
\mathbf{v}_m^{-1} (\mathbf{\bar{D}}' \mathbf{\bar{D}}/C^2_22).
\]

The homogenous solution of the problem is structured within the Trefftz formalism. The solution of the Laplace problem over \( \Gamma \cup \Omega \) is approximated by a linear combination of fundamental solutions, which are expressed in terms of sources located outside \( \Gamma \cup \Omega \). It is represented by the \( N_F \) global approximation functions \( \psi_i^f(p) \) and their coefficients \( \zeta_i \)
\[
\mathbf{d}(p) = \psi_i^f(p) \zeta_i
\]
\[
\text{i.e., they are fundamental solutions of the Laplace operator.}
This solution is for the two dimensional problems equal to
\[
\psi^*_N = \frac{1}{2\pi} \log \frac{r^0}{r_i}, \quad \psi^*_{N+1} = \psi^*_{N-1} = 1,
\]
\[
\psi^*_{N+2} = \psi^*_{N-2} = p_x - p_y, \quad \psi^*_{N+3} = \psi^*_{N+3} = p_y - p_y^0,
\]
(46)

with \(r^0\) standing for the reference radius and
\[
r_i = r_ir_i^r, \quad r_i = r^0\bar{r}_i, \quad r_g = p_g - s_g.
\]

The coefficients \(\zeta_i^*\) can be calculated by means of direct collocation or in the least squares sense [12]. In the former approach, a system with a number of linear equations equal to the number of unknowns is solved, and in the latter case, with a number of sources smaller or greater than the number of boundary nodes. For the sake of simplicity, the former approach is discussed in this paper. The coefficients \(\zeta_i^*\) are thus calculated from a system of \(N_f\) algebraic equations
\[
\Psi_{x}^* \zeta_{x}^* = b_{x},
\]
(48)
The first \(N_{fcol}\) equations are obtained through collocation of Eq. (44) in collocating points \(p_i\), distributed over \(\Gamma\)
\[
\psi_i^* = \psi_i^*(p_i), \quad \Psi_{x}^* = \chi_i^* \psi_i^0 + \chi^*_x \frac{\partial}{\partial n_{x}} \psi_i^0 + \chi_i^* \frac{\partial}{\partial n_{y}} \psi_i^0,
\]
(49)
\[
b_i = \chi_i^* (\tilde{Y}_i + \tilde{Y}_i^0 \Phi_i) + \chi_i^* (\tilde{Y}_i^0 + \tilde{Y}_i^0 \Phi_i)
\]
+ \(\chi_i^* (\tilde{Y}_i + \tilde{Y}_i^0 \Phi_i)\)
(50)
The remaining \(N_f - N_{fcol}\) equations are obtained through constraints
\[
\Psi_{y}^* = \psi_j^*(p_j); \quad j = N_{fcol}+1, N_{fcol}+2, \ldots, N_f,
\]
(51)
\[
b_j = 0; \quad j = N_{fcol}+1, N_{fcol}+2, \ldots, N_f,
\]
(52)
where we assumed
\[
\psi_j^*(p_j) = 0; \quad j = N_{fcol}+1, N_{fcol}+2, \ldots, N_f.
\]
(53)
The coefficients \(\zeta_i^*\) can be expressed through inversion of the system (48) that gives
\[
\zeta_i^* = \Psi_{x}^{-1} \left[ \chi_i^* (\tilde{Y}_i + \tilde{Y}_i^0 \Phi_i) + \chi_i^* (\tilde{Y}_i + \tilde{Y}_i^0 \Phi_i) \right]
\]
\[
+ \chi_i^* (\tilde{Y}_i + \tilde{Y}_i^0 \Phi_i)]
\]
(54)

3.5. The complete solution

The approximate solution to the complete problem can now be written in the following form
\[
\Phi(p) = \psi_i^*(p) \Psi_{x}^{-1} [\chi_i^* (\tilde{Y}_i + \tilde{Y}_i^0 \Phi_i) + \chi_i^* (\tilde{Y}_i + \tilde{Y}_i^0 \Phi_i)]
\]
\[
+ \chi_i^* (\tilde{Y}_i + \tilde{Y}_i^0 \Phi_i)]
\]
(55)
The previous equation can be written in the \(N_{col} - N_{fcol}\) collocation nodes \(p_k\), distributed over \(\Omega\) and \(\Gamma\) that do not belong to the Dirichlet boundary conditions. This gives us the following system of \(N_{col} - N_{fcol}\) collocation equations for solution of the unknown \(\Phi_i\) in collocation points \(p_i\), distributed over points that do not belong to the Dirichlet part of the boundary. The solution of the following system of algebraic equations completes the definition for the unknowns in \(\Omega\) and \(\Gamma\) boundary points
\[
\Psi_{x}^* \Phi_i = b_k;
\]
(56)

The simplest form of heat equation for temperature \(T\)
\[
\frac{\partial}{\partial t} (\rho c_p T) + \nabla \cdot (\rho c_p n T) = \nabla \cdot (k \nabla T)
\]
(59)

4. Numerical implementation

This chapter gives two simple examples how to implement the scalar valued and the vector valued transport equations in the present formulation and discusses other method-specific numerical implementation issues.

4.1. Energy transport—implementation into formulation
formulation as
\[ \Phi = c_p T, \quad \mathcal{F}(\Phi) = \Phi, \quad D = \frac{k}{c_p}, \quad D_{\xi\xi} = 0, \quad S = 0, \]
\[ \theta = \frac{\partial \mathcal{F}}{\partial k} \Phi - \frac{\partial \mathcal{F}}{\partial k} \Phi_0, \quad \theta_{,x} = \frac{\partial \mathcal{F}}{\partial k} \Phi, \]
\[ \Theta_y = \mathcal{F} v_x, \quad \Theta_{x,y} = \mathcal{F} v_y, \quad \Theta_{y,y} = \mathcal{F} v_y \]
(60)

4.2 Momentum transport—implementation into formulation

The simplest form of incompressible momentum transport of Newtonian fluid with Bussinesq term can be for velocity \( v \) written in the form
\[ \frac{\partial}{\partial t} \phi(v) + \nabla \cdot (\phi v) = -\nabla p + \nabla \cdot (\mu \nabla v) + \rho \mathbf{a} (1 - \beta (T - T_{\text{ref}})) \]
(61)
with pressure \( P \), constant viscosity \( \mu \), volumetric thermal expansion coefficient \( \beta \), acceleration \( \mathbf{a} \), and Bussinesq reference temperature \( T_{\text{ref}} \). The respective definitions in the formulation for the two momentum equations (the left equation is valid for the \( v_x \) component, the right equation is valid for the \( v_y \) component) are
\[ \phi = v_x, \quad \Phi = v_y \]
(62)
\[ \mathcal{F}(\Phi) = \Phi, \quad \mathcal{F}(\Phi) = \Phi \]
\[ D = \mu, \quad D_{t} = D \]
\[ D_{\xi} = 0, \quad D_{\xi \xi} = 0 \]
\[ S = -P_x + \rho \mathbf{a} (1 - \beta (T - T_{\text{ref}})), \quad S = -P_y + \rho \mathbf{a} (1 - \beta (T - T_{\text{ref}})) \]
\[ \theta = \frac{\partial}{\partial \mu} \Phi - \frac{\partial}{\partial \mu} \Phi_0 - \frac{S}{\mu}, \quad \theta = \frac{\partial}{\partial \mu} \Phi - \frac{\partial}{\partial \mu} \Phi_0 - \frac{S}{\mu} \]
\[ \theta_{,x} = \frac{\partial}{\partial \mu} \Phi, \quad \Theta_y = \frac{\partial}{\partial \mu} \Phi \]
\[ \Theta_y = \frac{\partial}{\partial \mu} \Phi, \quad \Theta_{x,y} = \frac{\partial}{\partial \mu} \Phi \]
\[ \Theta_{y,y} = \frac{\partial}{\partial \mu} \Phi \]

4.3 Numerical implementation issues

Three issues exist with respect to numerical implementation of the present method. The first one lies in the selection of radial basis function, the second is connected with the position of the nodes of the Trefftz method, and the third is the efficient treatment of the system (56) which has to be recalculated and solved in each iteration. The defined problems can be reasonably coped by the strategies that follow.

The numerical implementation can be based on various radial basis functions (first order conicals, third order conicals, multiquadrics, inverse multiquadrics, Gaussians, etc.) The following scaled augmented thin plate splines can be used in two dimensions
\[ \phi \phi(p) = r_n^3 \log r_n, \quad n = 1, 2, \ldots, N_{\text{col}}, \]
\[ \phi \phi(N_{\text{col}} + 1)(p) = 1, \]
\[ \phi \phi(N_{\text{col}} + 2)(p) = p_x - p_x^0, \]
\[ \phi \phi(N_{\text{col}} + 3)(p) = p_y - p_y^0, \]
with
\[ r_n^2 = (p - p_x)(p - p_y), \]
(64)
as a possible choice which does not involve any free parameter. Scaling constants \( p_x^0 \) and \( p_y^0 \) stand for the mean coordinates of the domain \( \Omega \). The adjacent harmonic functions \( \phi \phi \) for the thin plate splines (Eq. (63)) (the selection is not unique!) are
\[ \phi \phi = \frac{1}{16} r_n^4 \log r_n - \frac{1}{32} r_n^4; \quad n = 1, 2, \ldots, N_{\text{col}}, \]
\[ \phi \phi(N_{\text{col}} + 1) = \frac{1}{4} (p_x - p_x^0)^2 + \frac{1}{4} (p_y - p_y^0)^2, \]
\[ \phi \phi(N_{\text{col}} + 2) = \frac{1}{6} (p_x - p_x^0)^3, \]
\[ \phi \phi(N_{\text{col}} + 3) = \frac{1}{6} (p_y - p_y^0)^3. \]
The position of the nodes in the Trefftz method can be in the present context estimated by adopting the concept, very recently proposed by Saavedra and Power [9]. They use node position refinement procedure based on the statistical concept of significant parameters in multiple regression in connection with the least squares source position determination approach.

The full system (Eq. (56)) can be solved by the iterative techniques [10]. Many computational improvements to the described basic ideas are possible. For example use of the compactly supported radial basis functions with multilevel approach as described in Ref. [11] that would probably allow for solution of very large problems.

5. Conclusions

During the past three decades the humanity has heavily invested their resources in developing classical polygonization based discrete approximate methods. The present paper for the first time shows a mesh-free formulation for the general transport equation. The proposed formulation has
the following key features [13]: (i) It requires neither domain nor surface discretization. It is polygon-free. (ii) The formulation is similar for 2D and 3D problems. (iii) It does not involve numerical integration. (iv) Ease of learning. (v) Ease of coding. (vi) Competitiveness because of the man-power reduction involved for the meshing. The Laplace equation fundamental solution weighted and the augmented thin plate splines governed DRBEM has been in the last years applied to completely non-linear heat transfer and fluid flow problems. This include conductive–convective heat transport with non-linear material properties, non-linear boundary conditions, and phase change [14, 15], incompressible Navier–Stokes equations [16], and coupled problems [17] like natural convection [18] in Darcy porous media and in Newtonian fluid with and without phase-change [19]. The method found related industrial application in DC casting of aluminium alloy billets [20] and slabs [21]. The DRBEM and present method can both be theoretically developed from the method of particular solutions platform. The treatment of the particular solutions is in both methods equivalent, they differ only in treatment of the homogenous solution. In DRBEM, the homogenous solution is treated through Green’s formula and in the present method, the homogenous solution is treated by the global approximation with the fundamental solution. Both approaches have been theoretically well coped in the past. An comprehensive overview with selected bibliography can be found in Refs. [22,23]. The referenced spectra of problems will be recalculated by the represented polygon-free approach in one of our future publications.

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