Primitive variable dual reciprocity boundary element method solution of incompressible Navier–Stokes equations

Božidar Šarler\textsuperscript{a,\*}, Günther Kuhn\textsuperscript{b}

\textsuperscript{a}Laboratory for Fluid Dynamics and Thermodynamics, Faculty of Mechanical Engineering, University of Ljubljana, Aškerčeva 6, SI-1000 Ljubljana, Slovenia
\textsuperscript{b}Institute of Applied Mechanics, Technical Faculty, University Erlangen-Nuremberg, Egerlandstrasse 5, D-91058, Germany

Abstract

This paper describes the solution to transient incompressible two-dimensional Navier–Stokes equations in primitive variables by the dual reciprocity boundary element method. The coupled set of mass and momentum equations is structured by the fundamental solution of the Laplace equation. The dual reciprocity method is based on the augmented thin plate splines. All derivatives involved are calculated through integral representation formulas. Numerical example include convergence studies with different mesh size for the classical lid-driven cavity problem at Re \( \hat{\text{Re}} = 100 \) and comparison with the results obtained through calculation of the derivatives from global interpolation formulas. The accuracy of the solution is assessed by comparison with the Ghia–Ghia–Shin finite difference solution as a reference. © 1999 Elsevier Science Ltd. All rights reserved.

Keywords: Navier–Stokes equations; Primitive variables; Dual reciprocity boundary element method; Scaled augmented thin plate splines; Lid-driven cavity flow

1. Introduction

The boundary integral equation methods (BIEM) gained wide acceptance in solving partial differential equations related to the fluid flow problems. A comprehensive review of the related scientific and engineering achievements can be found in Ref. [1]. The BIEM [2] is an alternative numerical method for solving partial differential equations, characterized by choosing an appropriate fundamental solution as a kernel function of integral and by using the generalized Green’s formula for complete transfer of one or more partial differential operators on the kernel function. The main comparative advantage of the boundary element method over other discrete approximate methods is featured in cases where this procedure results in the boundary integral equation only. This turns out to be possible only for partial differential equations in some specific fluid flow.

The dual reciprocity boundary element method (DRBEM) [3] represents one of the possibilities for transforming the resultant domain integrals approximately into a finite series of boundary integrals. The key point of the DRBEM is approximation of the field in the domain by a set of global approximation functions and subsequent representation of the domain integrals of these global approximation functions by the boundary integrals. The DRBEM reduces all calculations to evaluation of the boundary integrals only. This fact might be advantageous in geometrically involved situations that are frequently encountered in fluid flow problems.

A broad spectrum of fluid flow phenomena are encountered in modelling solid–liquid phase change problems that range from creeping to turbulent flow. When we deal with melting or solidification of pure substances, at least mass, momentum and energy transfer have to be taken into account. In case of multicomponent systems, these equations have to be additionally accompanied by the species transfer.

Šarler and Kuhn recently developed [4,5] DRBEM solution to one-phase formulated convective–diffusive solid–liquid energy transport problems including nonlinear material properties and boundary conditions. Respectively, a great interest exists to enable the DRBEM for related fluid flow phenomena. Computational treatment of solid–liquid phase change flows within the one-phase formulation [6] should allow for variable density, viscosity and body force.

Owing to the physical complexity of the solid–liquid phase change flows, a prospective way to formulate the problem is in the primitive variables. A formulation for DRBEM solution of such flows coping the equiaxed and the columnar limits has been presented in Ref. [7]. The
present paper shows the first step in numerical implementation of related DRBEM formulation by concentrating on low Reynolds number flows with constant density, constant viscosity and omitted body force.

Very recently, Rahaim and Kassab [8] proposed a solution to coupled fluid-flow and heat transfer problem by the DRBEM constructed by using the fundamental solution of the Laplace equation. However, they showed solutions based on \(1 + r_x\) global approximation functions only for relative simple unidirectional flow in a tube.

The fluid flow treatment in this paper is similar to the one presented in the aforementioned paper for the following two respects: primitive variable context as well as the use of the same fundamental solution. It differs in the following principal points: transient character instead of the steady-state one, use of scaled augmented thin plate spline approximation functions instead of the \(1 + r_x\) ones, use of integral representation formula for the derivatives instead of derivation of the global interpolation functions.

The formulation used in the present paper could be roughly seen as the DRBEM implementation of the Grigor’ev scheme [9] with the domain cell integration replaced by the dual reciprocity formalism.

### 2. Governing equations

Consider a connected fixed domain \(\Omega\) with boundary \(\Gamma\) occupied by incompressible Newtonian fluid with constant viscosity \(\mu\). The governing equations for mass and linear momentum transport in such media are

\[
\nabla \cdot \mathbf{V} = 0
\]

\[
\rho \frac{\partial \mathbf{V}}{\partial t} + \rho \mathbf{V} \cdot \nabla \mathbf{V} = -\nabla P + \mu \nabla^2 \mathbf{V} \tag{2}
\]

where \(t\), \(\mathbf{V}\), and \(P\) represent time, velocity, and pressure, respectively.

The solution of the governing equations for the velocity and pressure field at final time \(t = t_0 + \Delta t\) is sought where \(t_0\) represents the initial time and \(\Delta t\) the positive time increment. The solution is constructed by the initial and boundary conditions that follow. The initial conditions in position \(\mathbf{p}\) at time \(t_0\) are

\[
\mathbf{V}(\mathbf{p}, t_0) = \mathbf{V}_0; \quad \mathbf{p} \in \Omega \tag{3}
\]

where \(\mathbf{V}_0\) represents a known function. Condition \(\nabla \cdot \mathbf{V}_0 = 0\) is assumed for a well posed problem. The boundary \(\Gamma\) is divided into not necessarily connected parts \(\Gamma^D\) and \(\Gamma^N\)

\[
\Gamma = \Gamma^D \cup \Gamma^N \tag{4}
\]

with Dirichlet and Neumann velocity boundary conditions respectively. These boundary conditions are at point \(\mathbf{p}\) and time \(t_0 \leq t \leq t_0 + \Delta t\) defined through known functions \(\mathbf{V}_f^D\) \(\mathbf{V}_f^N\)

\[
\mathbf{V}(\mathbf{p}, t) = \mathbf{V}_f^D; \quad \mathbf{p} \in \Gamma^D \tag{5}
\]

\[
\frac{\partial \mathbf{V}(\mathbf{p}, t)}{\partial n} = \mathbf{V}_f^N; \quad \mathbf{p} \in \Gamma^N
\]

### 3. Solution procedure

The elements of the solution procedure are explained in four entities. Discretization over time is explained first, followed by discussion of iterative procedure, and explanation of the discretization over space. Finally, the details of the numerical implementation are given.

#### 3.1. Discretization over time

Discretization of the momentum Eq. (2) over time is performed by fully implicit two-level finite differencing

\[
\rho \frac{\mathbf{V} - \mathbf{V}_0}{\Delta t} + \rho \mathbf{V} \cdot \nabla \mathbf{V} = -\nabla P + \mu \nabla^2 \mathbf{V} \tag{6}
\]

where index 0 denotes time \(t_0\) and no index \(t_0 + \Delta t\), respectively.

#### 3.2. Overview of the iteration cycle

The solution is performed in an iterative way. The discussion of one iteration cycle that follows explains calculation of the velocity and pressure fields at iteration \(m + 1\) from known velocity and pressure fields at iteration \(m\), subject to initial and boundary conditions.

The iteration cycle starts by evaluation of the velocity field at the boundary and domain nodes. This is followed by correction of the pressured field, correction of the velocity field, and correction of the velocity gradient fields. The iterations over timestep are completed when the condition

\[
||\mathbf{V}^{m+1}|| - ||\mathbf{V}^m|| < V_e
\]

is satisfied with \(V_e\) representing the velocity convergence criterion.

The differential form of the time discretized momentum equation is rewritten in boundary-domain integral form in order to be amenable for the DRBEM solution. For this purpose, Eq. (6) is weighted over the domain \(\Omega\) by the fundamental solution of the Laplace equation \(T^*\). The present paper is limited to two-dimensional Cartesian coordinates, e.g.

\[
T^* = \frac{1}{2\pi} \log \frac{\tilde{r}}{r} \tag{8}
\]

where \(\tilde{r}\) represents reference radius and \(r\) equals

\[
r = \mathbf{r} \cdot \mathbf{r}; \quad r = r_x i_x + r_y i_y; \quad r_x = p_x - s_x, \quad r_y = p_y - s_y
\]

\(p_x, p_y\) denote the Cartesian coordinates (base vectors \(i_x, i_y\)) of
point \( \mathbf{p} \), and \( s_i \), \( s_j \), the Cartesian coordinates of the fundamental solution source point \( \mathbf{s} \) respectively.

Green’s second theorem for vectors allows us to write the following boundary-domain integral version of weighted equation (6)

\[
\int_{\Gamma} \frac{\partial m^1 \hat{\mathbf{V}}}{\partial n} T^s d\Gamma - \int_{\Omega} m^1 \hat{\mathbf{V}} \frac{\partial T^s}{\partial n} d\Omega = c^* m^1 \hat{\mathbf{V}}
\]

where \( c^* \) stands for the fundamental solution related factor. Index \( \mathbf{s} \) denotes evaluation in the fundamental solution source point position. The source term in the velocity Poisson equation (10) equals

\[
\hat{S}_\mathbf{v} = \frac{1}{\mu} \left[ \frac{\partial m^1 \hat{\mathbf{V}} - \mathbf{V}_0}{\Delta r} + \frac{\partial m^1 \hat{\mathbf{V}} \cdot \nabla m^1 \mathbf{V} - \nabla^2 m^1 \mathbf{V}}{\Delta r} \right]
\]

Eq. (10) is simultaneously solved for unknown velocity \( m^1 \hat{\mathbf{V}} \) in the domain and unknown velocity \( m^1 \hat{\mathbf{V}} \) and velocity gradient \((\partial m^1 \hat{\mathbf{V}}/\partial n)\) on the boundary. The hat over the velocity field denotes that it generally does not satisfy the mass conservation equation. A correction \( m^1 \hat{\mathbf{V}} \) to \( m^1 \hat{\mathbf{V}} \) is assumed after solving Eq. (10)

\[
m^1 \mathbf{V} = m^1 \hat{\mathbf{V}} + m^1 \hat{\mathbf{V}}
\]

in such a way to force the mass conservation

\[
\nabla \cdot (m^1 \hat{\mathbf{V}} + m^1 \hat{\mathbf{V}}) = 0
\]

Consider that the velocity correction \( m^1 \hat{\mathbf{V}} \) occurs exclusively as a result of the action of the pressure gradient correction \( \hat{\mathbf{P}} \)

\[
\xi \frac{\partial}{\partial t} m^1 \hat{\mathbf{V}} = - \nabla m^1 \hat{\mathbf{P}}
\]

where \( \xi \) stands for the dimensionless velocity–pressure correction relaxation factor. The pressure correction could thus be calculated from the pressure Poisson equation

\[
\nabla^2 m^1 \hat{\mathbf{P}} = \frac{\partial}{\partial t} m^1 \hat{\mathbf{V}}
\]

deduced from Eqs. (13) and (14). The differential form of the pressure correction velocity Poisson equation (15) is rewritten in boundary-domain integral form in order to be amenable for the DRBEM solution. The boundary-domain integral expression for solving the pressure correction Poisson equation is

\[
\int_{\Gamma} \frac{\partial m^1 \hat{\mathbf{P}}}{\partial n} T^s d\Gamma - \int_{\Omega} m^1 \hat{\mathbf{P}} \frac{\partial T^s}{\partial n} d\Omega = c^* m^1 \hat{\mathbf{P}}
\]

\[
= \int_{\Omega} m^1 \mathbf{S}_p T^s d\Omega
\]

with the source term in the form

\[
m^1 \mathbf{S}_p = \xi \frac{\partial}{\partial t} \nabla \cdot m^1 \hat{\mathbf{V}}
\]

The boundary conditions of the pressure correction Poisson equation are

\[
\frac{\partial m^1 \hat{\mathbf{P}}}{\partial n} = m^1 \hat{\mathbf{P}}^N = 0; \quad \mathbf{p} \in I^D
\]

\[
m^1 \hat{\mathbf{P}} = m^1 \hat{\mathbf{P}}^N = 0; \quad \mathbf{p} \in I^N
\]

Note the Neumann boundary conditions for pressure correction at the points where the Dirichlet boundary conditions for the velocity are imposed and the Dirichlet boundary conditions for pressure correction at the points where the Neumann boundary conditions for velocity are imposed. The Neumann boundary conditions (18) at \( I^D \) of the boundary can be derived straightforward from Eq. (14) as

\[
\xi \frac{\partial}{\partial n} m^1 \hat{\mathbf{V}} \mathbf{t}_f = - \frac{\partial m^1 \hat{\mathbf{P}}}{\partial n}; \quad \mathbf{p} \in I^D
\]

The condition (19) originates from the fact that the velocity correction \( m^1 \hat{\mathbf{V}} \) equals 0 at the velocity Dirichlet boundary.

The pressure correction, pressure gradient correction, pressure and pressure gradient fields are obtained in the following way. The pressure correction Poisson equation (16) is first used to solve for pressure correction unknowns \( m^1 \hat{\mathbf{P}} \), \( m^1 \hat{\mathbf{P}}/\partial n \) on the boundary. Eq. (16) at internal points is subsequently used to explicitly solve the pressure correction unknowns \( m^1 \hat{\mathbf{P}} \) in the domain. The complete pressure field is updated as

\[
m^1 \mathbf{P} = m \mathbf{P} + m^1 \hat{\mathbf{P}}
\]

The pressure field is arbitrary up to an additive constant in the present incompressible fluid flow context. Pressure over the boundary and in the domain is in our case shifted by an additive constant \( P_c \) in order to be distributed around 0 (for example for plotting purposes)

\[
m^1 \mathbf{P} = m \mathbf{P} + m^1 \hat{\mathbf{P}}
\]

The constant \( P_c \) is determined from

\[
m^1 \mathbf{P}_c = - \frac{1}{\gamma} \int_{\Gamma} m^1 \mathbf{P} d\Gamma
\]

with \( \gamma \) standing for the total length of the boundary \( \Gamma \)

\[
\gamma = \int_{\Gamma} d\Gamma
\]

The pressure correction gradient \( m^1 \nabla \hat{\mathbf{P}} \) on the boundary and in the domain is explicitly calculated from the integral representation formula

\[
\int_{\Gamma} \frac{\partial m^1 \hat{\mathbf{P}}}{\partial n} \nabla T^s d\Gamma - \int_{\Omega} m^1 \hat{\mathbf{P}} \frac{\partial T^s}{\partial n} d\Omega = \nabla (c^* m^1 \hat{\mathbf{P}})
\]

\[
= \int_{\Omega} m^1 \mathbf{S}_p \nabla T^s d\Omega
\]

obtained by the action of the fundamental solution source
system gradient on the pressure correction Poisson equation Eq. (16). The complete pressure gradient field is updated as

$$\nabla^{m+1}p = \nabla^{m}p + \nabla^{m+1}\tilde{p}$$  (26)

The velocity field is updated through the pressure gradient corrections

$$m+1V = m+1\dot{V} - \frac{\Delta t}{\xi} \nabla^{m+1}\tilde{p}$$  (27)

The velocity gradients on the boundary and in the domain are explicitly calculated through the integral representation formula

$$\int_{\Gamma} \frac{\partial^{m+1}\dot{V}}{\partial t} d\Gamma - \int_{\Gamma} m+1V \frac{\partial \dot{T}}{\partial t} d\Gamma + \nabla(c^{m+1}V)_k = \int_{\Omega} m+1S \nabla \dot{T} d\Omega$$  (28)

that is obtained from Eq. (10) in the same way as Eq. (25) from Eq. (16). The normal derivatives of velocity on the boundary are not updated, so they are marked with a hat in the upper equation. The source $S\dot{V}$ in Eq. (28) differs from the source $S\dot{V}$, defined in Eq. (11) as a result of already updated pressure and velocity fields

$$m+1S \dot{V} = \frac{1}{\mu} \left[ \frac{m+1V - V_0}{\Delta t} + m+1\dot{V} \nabla \nabla^{m+1}p \right]$$  (29)

After calculation of the velocity field, checking Eq. (7) is performed. In case of convergence, a move to a new timestep is made. In this case the initial values of the new timestep are taken as the final values of the current timestep. When the convergence is not achieved, the velocity and its gradients are relaxed as

$$m+1V = m\dot{V} + \Xi (m+1V - m\dot{V})$$  (30)

$$\nabla^{m+1}V = \nabla m\dot{V} + \Xi (\nabla^{m+1}V - \nabla m\dot{V})$$  (31)

with $\Xi$ standing for the velocity relaxation factor, and the iteration procedure explained earlier makes another cycle. With this the discussion of the iteration procedure over timestep is complete.

3.3. Discretization over space

The simplest discretization is used to approximate the boundary integrals in Eqs. (10), (16), (25) and (28): Boundary geometry is approximated by $N_\Gamma$ straight line segments, and spatial variation of the fields on each of the boundary segments is represented by constant interpolation functions with gridpoints coinciding with the geometrical centers of the straight line segments. The Einstein summation convention is used in this text, i.e. any index which is repeated twice in a product is summed up. An underlined index is not summed up. For boundary integrals of a scalar valued function, or boundary integrals of a component of a vector valued function, both denoted by the function $\mathcal{F}$, this gives

$$\int \frac{\partial \mathcal{F}}{\partial t} T_i d\Gamma - \int \frac{\partial T_i}{\partial t} \nabla \mathcal{F} = G_i \delta_{ij} \frac{\partial \mathcal{F}}{\partial t} - H_i \delta_{ij} \mathcal{F} = c_i^* \delta_{ij} \mathcal{F},$$  (32)

where $k = 1,2,\ldots, N_\Gamma$ and $i, l = 1,2,\ldots, N$. $N = N_\Gamma + N_\Omega$ is the total number of points in which the solution is calculated. The first $N_\Gamma$ points $p_k$ coincide with the boundary gridpoints and the last $N_\Omega$ points are arbitrarily distributed over the domain $\Omega$. Index $l$ stands for $s_l = p_v$. $\delta$ represents the Kronecker symbol. Matrix elements $G_{ik}$ and $H_{ik}$ are defined as follows.

$$G_{ik} = \int \frac{\partial T_i}{\partial t} T_j d\Gamma, \quad H_{ik} = \int \frac{\partial T_i}{\partial t} \nabla \mathcal{F} d\Gamma,$$  (33)

where $\Gamma_k$ represents the $k$-th boundary segment, and $c_i^*$ is equal to

$$c_i^* = 0; s_l \notin \Omega \cup \Gamma, \quad c_i^* = \frac{1}{2}; s_l \in \Gamma,$$  (34)

$$c_i^* = 1; s_l \in \Omega.$$

The domain integrals in Eqs. (10) and 16) are transformed by considering the approximation of the spatial variation of the fields in $\Omega$ by the global interpolation functions of the form

$$\mathcal{F}(p) = \psi_k(p) \xi_n, u = 1,2,\ldots,N + 3.$$  (35)

The two-dimensional scaled augmented thin plate splines are used in this work, because of theoretical considerations, described in Ref. [10]

$$\psi_n(p) = r_n^2 \log r_n; \quad n = 1,2,\ldots,N$$  (36)

$$\psi_{N+1}(p) = p_x - p_x^0$$

$$\psi_{N+2}(p) = p_y - p_y^0$$

$$\psi_{N+3}(p) = 1$$ with

$$r_n^2 = (p - p_x^0)(p - p_y^0)$$  (37)

Scaling constants $p_x^0$ and $p_y^0$ stand for the mean coordinates of the domain $\Omega$. Coefficients $\xi_n$ are calculated from a system of $N + 3$ algebraic equations

$$\Psi_s = \mathcal{F}$$  (38)

where $s = (s_1, s_2,\ldots, s_{N+3})^T$ and $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2,\ldots, \mathcal{F}_N, 0, 0, 0)^T$. The first $v = 1,2,\ldots,N$ rows of matrix $\Psi$ are of the form $(\psi_{s_1}, \psi_{s_2},\ldots, \psi_{s_{N+3}})$, and the last three rows $v = N + 1, N + 2, N + 3$ are of the form $(\psi_{s_1}, \psi_{s_2},\ldots, \psi_{s_{N+3}}, 0,0,0)$, where the notation has been shortened to $\Psi_s = \mathcal{F}(p_n), \psi_{mn} = \psi_s(p_n)$.  

Coefficients \( c_n \) are given by
\[
s_n = \mathbf{\Psi}^{-1}\mathbf{F}
\]  

Consequently, the domain integral can be written in a compact dual reciprocity form \((k = 1, 2, \ldots, N, \ i, l = 1, 2, \ldots, N + 3)\)
\[
\int_\Omega \mathbf{F}^T_l d\Omega = \int_\Omega \psi_u \mathbf{\Psi}^{-1}\mathbf{F}^T_l d\Omega = \mathbf{\Psi}_{lu} \mathbf{\Psi}^{-1}\mathbf{F}_l,
\]  

with
\[
\mathbf{\Psi}_{lu} = \int_\Omega \psi_u T_l d\Omega
\]  

The integral \( \mathbf{\Psi}_{lu} \) is calculated by defining the harmonic functions \( \hat{\psi}_u \)
\[
\nabla^2 \hat{\psi}_u(p) = \psi_u(p)
\]  

which allow us to write
\[
\mathbf{\Psi}_{lu} = \int_\Gamma \frac{\partial \hat{\psi}_u}{\partial n}\mathbf{T}^s d\Gamma - \int_\Gamma \hat{\psi}_u \frac{\partial \mathbf{T}^s}{\partial n} d\Gamma - c_l^2 \hat{\psi}_u(s)
\]  

The aforementioned integrals are numerically evaluated by using the same discretization strategy that leads to expression Eq. (32)
\[
\int_\Omega \nabla^2 \hat{\psi}_u T_l d\Omega = G^\phi_{kl} \frac{\partial \hat{\psi}_u}{\partial n} - H^\phi_{kl} \hat{\psi}_u - c_l^2 \delta_{l}\hat{\psi}_u
\]  

However, matrices \( G^\phi \) and \( H^\phi \) differ from matrices \( G \) and \( H \) because the boundary subdomain that corresponds to the calculation of the fields on the boundary and integrals \( \mathbf{\Psi}_{lu} \) could differ in general. Let us denote the number of boundary segments leading to calculation of matrices \( G^\phi \) and \( H^\phi \) with \( N^\phi \). Therefore index \( k \) in Eq. (44) runs as \( k = 1, 2, \ldots, N^\phi \).

The adjacent harmonic functions \( \hat{\psi}_u \) for the thin plate splines Eq. (36) used (the selection is not unique!) in the present work are
\[
\hat{\psi}_n = \frac{1}{16} r_n^4 \log r_n - \frac{1}{32} r_n^4
\]
\[
\hat{\psi}_{N+1} = \frac{1}{6}(p_x - p_x^0)^3
\]
\[
\hat{\psi}_{N+2} = \frac{1}{6}(p_y - p_y^0)^3
\]
\[
\hat{\psi}_{N+3} = \frac{1}{4}(p_x - p_x^0)^2 + \frac{1}{4}(p_y - p_y^0)^2
\]  

The boundary integrals in Eqs. (25) and (28) are calculated as
\[
\int_\Gamma \frac{\partial \mathbf{F}}{\partial p_\xi} - \mathbf{T}_l^* d\Gamma = \int_\Gamma \frac{\partial \mathbf{F}}{\partial p_\zeta} - \mathbf{T}_l^* d\Gamma - c_l^2 \frac{\partial \mathbf{F}}{\partial p_\zeta}
\]  

with \( \zeta \) standing for \( x \) or \( y \) and matrix elements \( G_{\zeta l} \) and \( H_{\zeta l} \) are defined as follows:
\[
G_{\zeta l} = \int_\Gamma \frac{\partial}{\partial p_\zeta} T_l^* d\Gamma, \quad H_{\zeta l} = \int_\Gamma \frac{\partial^2 T_l^*}{\partial p_\zeta^2} d\Gamma
\]  

The domain integrals in Eqs. (25) and (28) are calculated as
\[
\int_\Omega \mathbf{F}^T \frac{\partial}{\partial p_\zeta} T_l^* d\Omega = \int_\Omega \psi_u \mathbf{\Psi}^{-1}\mathbf{F}^T_l d\Omega = \mathbf{\Psi}_{lu} \mathbf{\Psi}^{-1}\mathbf{F}_l
\]  

with
\[
\mathbf{\Psi}_{lu} = \int_\Omega \psi_u \frac{\partial}{\partial p_\zeta} T_l d\Omega.
\]  

The aforementioned integrals are evaluated similarly to those in Eq. (44)
\[
\int_\Omega \nabla^2 \hat{\psi}_u \frac{\partial}{\partial p_\zeta} T_l d\Omega = G^\phi_{kl} \frac{\partial \hat{\psi}_u}{\partial n} - H^\phi_{kl} \hat{\psi}_u - c_l^2 \delta_{l}\hat{\psi}_u
\]  

with index \( k \) running as \( k = 1, 2, \ldots, N^\phi \).

The diagonal elements of the twelve boundary element matrices \( G, H, G^\phi, H^\phi, G_{\zeta l}, H_{\zeta l}, G^\phi_{\zeta l}, H^\phi_{\zeta l} \) are calculated analytically in the present work, the others by 20-point Gaussian quadrature. Reference radius \( \bar{r} \) is set to 1. Relatively complex expressions for analytical evaluation of all matrix elements, that substantially speeds up the CPU time, are given in the complementary work [11].

### 3.4. Systems of algebraic equations

#### 3.4.1. Calculation of velocity

The first two systems of algebraic equations in the present context occur when we evaluate the unknown velocity components at the boundary and domain nodes and unknown normal velocity gradient components at the boundary nodes. After time and space discretization of Eq. (10) as described, the following compact form is obtained
\[
\mathbf{\Psi}_{lu}^{m+1} \hat{\psi}_u + \mathbf{\Psi}_{li} \frac{\mathbf{\Psi}_l^{m+1} \hat{\psi}_u}{\partial n} = \mathbf{\Psi}_{li}^{m+1} S_{VV;}, \quad i = 1, 2, \ldots, N
\]  

with the coefficients
\[
\mathbf{\Psi}_{li} = -H_{\zeta l} \delta_{li} - c_\zeta \delta_{li}
\]
\[
\mathbf{\Psi}_{li} = G_{\zeta l} \delta_{li}
\]
\[
\mathbf{\Psi}_{li} = \mathbf{\Psi}_{lu} \mathbf{\Psi}^{-1}
\]

Application of the boundary conditions (5) yields the following two systems of algebraic equations for calculating...
the $N$ unknowns at the boundary and in the domain

$$\mathbf{A}_t \mathbf{x}_t = \mathbf{b}_t$$  \hspace{1cm} (55)

with the systems matrix $\mathbf{A}_t$ in the form

$$A_{ti} = \chi_i \left[ \Psi_{ji} + \xi_i \mu \left( -\frac{1}{\Delta t} + \frac{\partial^m V_{\xi}^t}{\partial \xi_i} \right) \right] + (1 - \chi_i) \mathcal{B}_{ti};$$  \hspace{1cm} (56)

$i = 1, 2, \ldots, N$

The vector of unknowns $\mathbf{x}_t$ is

$$\mathbf{x}_t = \chi_i^{m+1} \hat{\mathbf{V}}_t + (1 - \chi_i) \frac{\partial^{m+1} \hat{\mathbf{V}}_t}{\partial \xi_i}; \hspace{1cm} i = 1, 2, \ldots, N$$  \hspace{1cm} (57)

and the right hand side vector $\mathbf{b}_t$ is

$$b_{ti} = \xi_i \mu \left[ -\frac{1}{\Delta t} V_{ti} + m \frac{\partial^{m+1} \hat{\mathbf{V}}_t}{\partial \xi_i} + \frac{\partial^m \mathcal{P}_t}{\partial \xi_i} \right]$$

$$- (1 - \chi_i) \left[ \Psi_{ji} + \xi_i \mu \left( -\frac{1}{\Delta t} + \frac{\partial^m V_{\xi}^t}{\partial \xi_i} \right) \right] V_{ti}^d$$

$$- \chi_i \mathcal{B}_{ti} V_{N}^d, \hspace{1cm} i = 1, 2, \ldots, N$$  \hspace{1cm} (58)

The boundary conditions indicator $\chi_i$ is equal to 0 for $\mathbf{p}_t \in \Gamma^b$ and equal to 1 for $\mathbf{p}_t \in \partial \Omega$. Index $\xi$ could not take the same value as index $\zeta$ in Eq. (58). The velocity gradient components at the boundary and domain nodes can be calculated from the following explicit formula

$$\frac{\partial^{m+1} \mathbf{V}}{\partial \xi_i} = -\frac{1}{c_t} \left[ \Psi_{\xi_t} \Psi_{w_i} \hat{\mathbf{V}}_t^{m+1} S_{\xi_t} - G_{\xi_t} \frac{\partial^{m+1} \hat{\mathbf{V}}_t}{\partial \xi_i} + H_{\xi_t} \frac{\partial^{m+1} \hat{\mathbf{V}}_t}{\partial \xi_i} \right]$$  \hspace{1cm} (59)

after the two algebraic equation systems Eq. (55) are solved.

3.4.2. Calculation of pressure correction

After the space discretization of Eq. (10) as described, the following compact form is obtained

$$\Psi_{hi} \hat{\mathbf{P}}_i + \Psi_{ri} \frac{\partial^{m+1} \hat{\mathbf{P}}}{\partial \xi_i} = \mathcal{C}_i$$  \hspace{1cm} (60)

with the coefficients defined as

$$\Psi_{hi} = -H_{\xi_t} \delta_{hi} - c_t \delta_{hi}$$  \hspace{1cm} (61)

$$\Psi_{ri} = G_{\xi_t} \delta_{hi}$$  \hspace{1cm} (62)

$$\mathcal{C}_i = \Psi_{\xi_t} \Psi_{w_i} \hat{\mathbf{V}}_t^{m+1} S_{\xi_t}.$$  \hspace{1cm} (63)

Application of the pressure correction boundary conditions Eq. (19) yields the following system of $N_\Gamma$ algebraic equations for unknown pressure corrections and unknown normal pressure correction gradients at the boundary

$$\mathbf{A}_p \mathbf{x}_p = \mathbf{b}_p$$  \hspace{1cm} (64)

with the system matrix $\mathbf{A}$ in the form

$$A_{pi} = (1 - \chi_i) \Psi_{ji} + \chi_i \mathcal{B}_{ti};$$  \hspace{1cm} (65)

The vector of unknown $\mathbf{x}$ is

$$x_i = (1 - \chi_i) \hat{\mathbf{P}}_i + \chi_i \frac{\partial^{m+1} \hat{\mathbf{P}}}{\partial \xi_i}; \hspace{1cm} i = 1, 2, \ldots, N_\Gamma$$

and the right hand side vector $\mathbf{b}$ is

$$b_i = \mathcal{C}_i - \chi_i \Psi_{ji} \frac{\partial^{m+1} \hat{\mathbf{P}}}{\partial \xi_i} + H_{\xi_t} \frac{\partial^{m+1} \hat{\mathbf{P}}}{\partial \xi_i}$$  \hspace{1cm} (66)

Pressure correction at the domain nodes is calculated from the following explicit formula

$$\frac{\partial^{m+1} \hat{\mathbf{P}}}{\partial \xi_i} = -\frac{1}{c_t} \left[ \mathcal{C}_i - \Psi_{ji} \frac{\partial^{m+1} \hat{\mathbf{P}}}{\partial \xi_i} + H_{\xi_t} \frac{\partial^{m+1} \hat{\mathbf{P}}}{\partial \xi_i} \right]$$  \hspace{1cm} (67)

after the algebraic equation system Eq. (64) is solved previously. Pressure correction gradient components at the boundary nodes and at the domain nodes are calculated from the following explicit formula

$$\frac{\partial^{m+1} \hat{\mathbf{P}}}{\partial \xi_i} = -\frac{1}{c_t} \left[ \Psi_{\xi_t} \Psi_{w_i} \hat{\mathbf{V}}_t^{m+1} S_{\xi_t} - G_{\xi_t} \frac{\partial^{m+1} \hat{\mathbf{P}}}{\partial \xi_i} + H_{\xi_t} \frac{\partial^{m+1} \hat{\mathbf{P}}}{\partial \xi_i} \right]$$  \hspace{1cm} (68)

A DRBEM code has been coded in NDP 4.6.0-4.4.1-1.9.0 FORTRAN with double precision accuracy. Test cases have been run on a 64 Mb memory Pentium-II 200 MHz personal computer with MS-DOS 6.22. The systems of linear equations have been solved by the standard Gaussian elimination with rows and columns pivoting [13]. The parameter $N_\Gamma$ has been set to $N_\Gamma$ in all computations. All numbers shown in Tables 1 and 2 have rounded last digit.

4. Numerical example

The main aim of numerical studies in this paper is to

<table>
<thead>
<tr>
<th>Solution</th>
<th>(a)</th>
<th>(b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Errmea/100 [m/s] Mesh I</td>
<td>0.0294</td>
<td>0.0347</td>
</tr>
<tr>
<td>Errmea/100 [m/s] Mesh II</td>
<td>0.0834</td>
<td>0.0966</td>
</tr>
<tr>
<td>Errmax/100 [m/s] Mesh I</td>
<td>0.0934</td>
<td>0.0966</td>
</tr>
<tr>
<td>Errmax/100 [m/s] Mesh II</td>
<td>0.0934</td>
<td>0.0966</td>
</tr>
<tr>
<td>Errmax/100 [m/s] Mesh II</td>
<td>0.0190</td>
<td>0.0195</td>
</tr>
</tbody>
</table>

Table I from [12].
Error max and Errmea are based on values in Table II from [12].

investigate the consistency, convergence, stability and robustness of the developed method. An appropriate benchmark often used in computational fluid dynamics is the square cavity flow with a driven upper wall [12].

Such a flow was calculated by the developed DRBEM technique under the following dimensional assumptions: Calculation domain $\Omega$ is a square with the coordinates $p_x < p_x < p_x^+$, $p_y < p_y < p_y^+$; $p_x^+ = p_y^+ = \pm 0.5 [m]$. The boundary $\Gamma$ is divided into four parts. $\Gamma_1$: $p_x^+ = p_y^+$, $p_x^+ = p_x^-$, north $\Gamma_N$: $p_y = p_y^+$, $p_x^+ = p_x^-$, east $\Gamma_E$: $p_x = p_x^+$, $p_y < p_y^-$, and west $\Gamma_W$: $p_x = p_x^-$, $p_y < p_y^+$ parts. The Dirichlet velocity boundary conditions are set to

\[
\begin{align*}
V_{T_1}^0 & = 0 [m/s]; \quad p \in \Gamma_S \cup \Gamma_E \cup \Gamma_W \\
V_{T_1}^0 & = 0 [m/s]; \quad p \in \Gamma_N \\
V_{T_1}^0 & = 100 [m/s]; \quad p \in \Gamma_N \\
V_{T_1}^0 & = 0 [m/s]; \quad p \in \Gamma_N \\

V_0 & = 0 [m/s]; \quad p \in \Omega + \Gamma \\
V_0 & = 0 [m/s]; \quad p \in \Omega + \Gamma \\

Initial pressure and pressure gradients are
\end{align*}
\]

\[
\begin{align*}
P_x & = 0 [N/m^2]; \quad p \in \Omega + \Gamma \\
\frac{\partial P}{\partial p_x} & = 0 [N/m^2]; \quad p \in \Omega + \Gamma \\
\frac{\partial P}{\partial p_y} & = 0 [N/m^2]; \quad p \in \Omega + \Gamma \\

Material properties are set to $\rho = 1 [kg/m^3]$, $\mu = 1 [kg/(m*s)]$ and the corresponding Reynolds number $Re = \rho V_0^2 (\Gamma_1) / \mu$ of the problem is thus 100.

The problem has been run on a uniform Mesh I with $N_T = 40, N_S = 81, N = 121$, and on a uniform Mesh II with $N_T = 80, N_S = 361, N = 441$. Both meshes are shown in Fig. 1. The relaxation coefficients $\xi$ and $\Xi$ have been both set to 1.

The time step used is $\Delta t = 0.001 [s]$. The velocity convergence criterion $V_e$, evaluated in all gridpoints, has been set to 0.001. The steady-state was assumed to be attained when the condition

\[
||V|| - |V_0| < V_\text{steady}
\]

is satisfied at any of the gridpoints. The value for $V_\text{steady}$ was taken as 0.01. The solution with Mesh I reaches a steady-state in 75 timesteps with the total number of 1035 internal timestep iterations and the maximum of 174 internal timestep iterations. The solution with Mesh II reaches a steady-state in 94 timesteps with the total number of 1602 internal timestep iterations and the maximum of 120 internal timestep iterations. The solutions based on both meshes are shown in Fig. 2. In Figs. 3 and 4 calculated velocity field is compared with the reference solution from [12]. As could be seen from the Figs. 3 and 4 our numerical method tends towards the reference solution with finer mesh.

The minimum value of the stream-function has been evaluated through formulas in Appendix A. The Mesh I gives stream-function minimum $-7.0686 [m^2/s]$, and the Mesh II gives the stream-function minimum $-9.2491 [m^2/s]$ which is closer to the value of $-10.3423 [m^2/s]$ in Ref. [12].

The derivatives of pressure and velocity have been in the present work calculated through the integral representation formulas (59) and (67). In work [8] these derivatives have been calculated through the derivatives of global interpolation, i.e.

\[
\begin{align*}
\mathbf{V}^{n+1} &= \nabla \psi_{\theta} \Psi_{-1}^{n+1} V_i \\
\mathbf{P}^{n+1} &= \nabla \psi_{\theta} \Psi_{-1}^{n+1} P_i
\end{align*}
\]

The numerical solution with calculation (59) replaced by Eq. (72) and with calculation (68) replaced by Eq. (73) has been numerically implemented as well. Tables I and 2 show comparison between the two solutions. Errmax has been defined as the maximum absolute difference between the calculated and reference value of the field. Errmax has been defined as the mean absolute difference between the calculated and the reference value of the field. Errmax and Errmea have been evaluated based on reference values from [12].

In case the derivatives of pressure and velocity are calculated through the global interpolation approximation the convergence properties are as follows: The solution with Mesh I reaches a steady-state in 75 timesteps with the total number of 917 internal timestep iterations and the maximum of 329 internal timestep iterations. The solution with Mesh II reaches a steady-state in 63 timesteps with the total number of 620 internal timestep iterations and the maximum of 78 internal timestep iterations.

The use of global interpolation representation formulas for velocity gradients and pressure correction gradients speeds-up the convergence. However, the solution is less accurate compared to use of the integral representation formulas. Even more accurate results are expected with higher order boundary elements.

Table 2
Accuracy of the two solutions in terms of $V_e$ along horizontal line through geometric center of the cavity. (a) derivatives represented by the integral representation formulas (59) and (67), (b) derivatives calculated through the global interpolation functions (72) and (73). Errmax and Errmea are based on data in Table II from [12].
Fig. 1. (a) Schematics of the DRBEM Mesh I. Boundary and domain mesh-points are represented with ● and ○ respectively. Boundary element borders are marked with ×. (b) Schematics of the DRBEM Mesh II. Same symbols as in Fig. 1a.
Fig. 2. (a) Calculated velocity field with MESH I. (b) Calculated velocity field with Mesh II.
Fig. 3. (a) $V_x$ along vertical line through geometric center of the cavity calculated with Mesh I. Reference values from [12] are marked with $\times$. (b) $V_x$ along vertical line through geometric center of the cavity calculated with Mesh II. Reference values from [12] are marked with $\times$. 
Fig. 4. (a) $V_y$ along horizontal line through geometric center of the cavity calculated with Mesh I. Reference values from [12] are marked with +. (b) $V_y$ along horizontal line through geometric center of the cavity calculated with Mesh II. Reference values from [12] are marked with +.
5. Conclusions

This paper describes the first attempt in solving the Navier–Stokes equations in recirculating flow situations by using DRBEM structured with the fundamental solution of the Laplace equation and scaled augmented thin plate splines. Results are obtained for the driven cavity flow at Re = 100. Despite the relatively rough mesh and most simple boundary elements the results show convergence and high accuracy. It has been found out that the use of integral representation formula for the derivatives gives better solution than the one with the derivatives calculated through global interpolation functions on the expense of more iterations.

Our future research will follow directions towards numerical implementation improvements and towards physically more involved situations. In the first direction, higher boundary elements will be numerically implemented. Their influence on the accuracy of the solution will be investigated. Higher Reynolds number flow situations will be attempted together with the sensitivity of timestep length and introduced relaxation factors on the number of required iterations. Since higher Reynolds number flow situations will require more dense mesh, the iterative solution of the equations will require more iterations. Since higher Reynolds number flow situations will be investigated. Higher Reynolds number flow situations will be attempted together with the sensitivity of timestep length and introduced relaxation factors on the number of required iterations. Since higher Reynolds number flow situations will require more dense mesh, the iterative solution of the equations will require more iterations.

Variation of the velocity components over the domain \( \Omega \) and boundary \( \Gamma \) is based on global approximation functions

\[
V_x = \psi_n \Psi^{-1} V_{xi}
\]

\[
V_y = \psi_n \Psi^{-1} V_{yi}
\]

Respectively, stream-function could be analytically calculated as

\[
\psi = \int_{p_{min}}^{p_f} \psi_0 dp_x \Psi^{-1} V_{xi}
\]

\[
\psi = -\int_{p_{min}}^{p_f} \psi_0 dp_y \Psi^{-1} V_{yi}
\]

The involved integrals over \( p_x \) are for \( \psi_1, \psi_2, \ldots, \psi_N \) analytically expressed as

\[
I_{tn} = I_{tn}(p_{x1}, p_{x2}) = \int_{p_{x1}}^{p_{x2}} \psi_0 dp_x;
\]

\[
x_{p_{x1}}, x_{p_{x2}} \leq p_{x} \quad \text{or} \quad x_{p_{x1}}, x_{p_{x2}} = p_{x}.
\]

\[
I_{tn} = I_{tn}(p_{x1}, p_{x2}) + I_{tn}(p_{x2}, p_{x3}) \quad p_{x1} \leq p_{x2} \leq p_{x3}
\]

\[
I_{tn} = \pm I_{tn}(p_{x1}, p_{x3}) \pm I_{tn}(p_{x1}, p_{x2})
\]

where the plus sign is used for \( p_{x1}, p_{x2} > p_{x} \) and the minus sign is used for \( p_{x1}, p_{x2} < p_{x} \). The value of integrals \( I_{tn} \) and \( I_{tn} \) is

\[
I_{tn} = \left[ r^2 - (p_y - p_x)^2 \right]^{1/2} r^2 \log r
\]

\[
- \frac{1}{3} \left( r^2 - (p_y - p_{x2})^2 \right)^{3/2} r(1 + 2 \log r) \bigg|_{r=r_2},
\]

\[
2I_{tn} = \left[ \frac{2}{3} \left( \frac{1}{3} x'^3 - |p_y - p_{x2}|^2 x' + |p_y - p_{x2}| \right) \right]^x'=p_{y1}-p_{y0}
\]

with

\[
r_1 = ((p_{x1} - p_{x0})^2 + (p_y - p_{x2})^2)^{1/2}
\]

\[
r_2 = ((p_{x2} - p_{x0})^2 + (p_y - p_{x2})^2)^{1/2}
\]

The involved integrals over \( p_y \) are for \( \psi_{N+1}, \psi_{N+2}, \psi_{N+3} \)

Acknowledgements

The authors would like to acknowledge the Ministry of Science and Technology of the Republic of Slovenia (MZT), and the German Ministry of Education and Research (BMBF) for support in the framework of the Slovene-German bilateral scientific and technological cooperation, project Solidification Modelling by the Boundary Element Method.

Appendix A

A.1. Calculation of stream-function

The velocity components are calculated from the stream-function \( \psi \) as

\[
V_x = \frac{\partial}{\partial p_y} \psi
\]

\[
V_y = -\frac{\partial}{\partial p_x} \psi
\]

The stream-function is calculated from the velocity components as

\[
\psi = \int_{p_{min}}^{p_f} V_x dp_y
\]

\[
\psi = -\int_{p_{min}}^{p_f} V_y dp_x
\]
analytically expressed as

\[ I_{N+1} = \int_{p_x}^{p_x'} \psi_{N+1} dp_x = \frac{(p_x - p_x')^2}{2} - p_0^2(p_x - p_x') \]

\[ I_{N+2} = \int_{p_x}^{p_x'} \psi_{N+2} dp_x = (p_x - p_x')(p_x - p_x') \]

\[ I_{N+3} = \int_{p_x}^{p_x'} \psi_{N+3} dp_x = p_x - p_x' \]

The relations for integrals over \( p_y \) coordinate could be easily perceived from the given formulae for the integrals over \( p_x \) coordinate and are thus not explicitly represented.

References


