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A generalized Exner equation for sediment mass balance

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[1] The advance of morphodynamics research into new areas has led to a proliferation of forms of sediment mass balance equation. Without a general equation it is often difficult to know what these problem-specific versions of sediment mass balance leave out. To address this, we derive a general form of the standard Exner equation for sediment mass balance that includes effects of tectonic uplift and subsidence, soil formation and creep, compaction, and chemical precipitation and dissolution. The complete equation, (17), allows for independent evolution of two critical interfaces: that between bedrock and sediment or soil and that between sediment and flow. By eliminating terms from the general equation it is straightforward to derive mass balance equations applicable to a wide range of problems such as short-term bed evolution, basin evolution, bedrock uplift and soil formation, and carbonate precipitation and transport. Dropping terms makes explicit what is not being considered in a given problem and can be done by inspection or by a formal scaling analysis of the terms. Scaling analysis leads directly to dimensionless numbers that measure the relative importance of terms in the equation, for example, the relative influence of spatial versus temporal changes in sediment load on bed evolution. Combining scaling analysis with time averaging shows how the relative importance of terms in the equation can change with timescale; for example, the term representing bed evolution due to temporal change in sediment load tends to zero as timescale increases.

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1. Introduction

[2] An analysis of sediment mass balance is fundamental to solving a wide range of problems in morphodynamics. The relations in common use for expressing sediment mass balance take a variety of forms but are all descended from the equation initially presented by Exner and reproduced below. The additions and changes that have been made to Exner's original equation have mostly been done piecemeal with the aim of adapting it for a particular problem. A summary of the Exner equation including forms appropriate for channelized systems and sediment mixtures is presented by *Parker* [2005], and detailed derivations of mass balance for soil-mantled hillslopes are presented by *Anderson* [2002] and *Mudd and Furbish* [2004]. Both of the latter works also explicitly include chemical effects. Nonetheless, each of these mass balance equations includes some terms and leaves out others, according to the problem at hand. Here we revisit the Exner equation with the aim of providing a complete general form. This generality allows the equation to be specialized for application to a wide range of morphodynamic problems by dropping or combining terms. We hope that the proposed general form will be especially useful for geologic problems for which processes such as tectonic uplift and subsidence, soil formation and creep, and dissolution and precipitation become important.

[3] It is easy to overlook the fact that a model is defined as much by what it leaves out as by what it includes. Using a general equation as a starting point makes this explicit: we develop specialized relations for specific problems by eliminating terms from the general equation. Elimination of terms can be done by inspection or by formal scaling analysis. We give examples of how to estimate the magnitudes of different terms below, after deriving the general mass balance equation. Because surface evolution occurs on timescales from seconds to millions of years, we then investigate the behavior of the general equation under time averaging. We conclude the paper by returning to one of the original motivations for it: the satisfaction of finding unity in apparently diverse phenomena. We hope that the examples that conclude the paper, showing how the general mass balance equation can be specialized to a variety of problems from classical morphodynamics to basin dynamics and carbonate precipitation, will illustrate how sediment mass balance can serve as one such unifying theme in surface dynamics.

2. Exner's Equation

[4] Felix Exner was a Viennese meteorologist who worked on a variety of topics in natural science. He

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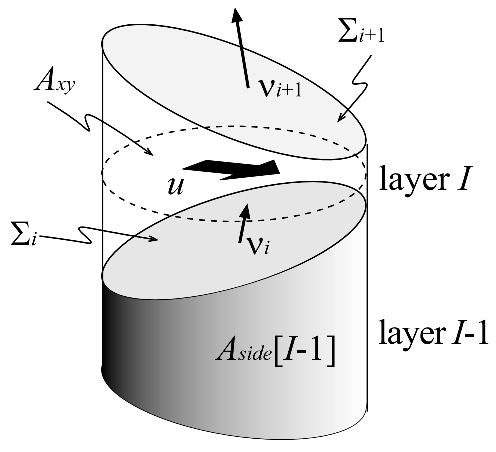

Figure 1. Definition sketch for mass balance in a layer with arbitrary top and bottom boundaries.

developed his equation for sediment mass balance as part of a pair of remarkably sophisticated studies of river morphology [*Exner*, 1920, 1925]. The equation was brought to the attention of the English-speaking world via the book by *Leliavsky* [1955]. The original equation given by Exner can be written as

$$\frac{\partial \eta}{\partial t} = -A\frac{\partial U}{\partial x} \tag{1}$$

where η is bed elevation relative to some fixed datum, t is time, A is a coefficient, U is average flow velocity, and x is downstream distance.

[5] In the text preceding the equation, Exner made clear that he intended the flow velocity U as a proxy for the sediment flux. Thus the "Exner equation" is now written with the sediment flux in place of the flow velocity. In addition, it has become standard to include a term representing temporal changes in sediment concentration. We will provide the standard modern form of the Exner equation below as a special case of a general mass balance equation.

3. General Derviation of Sediment Mass Balance

[6] It is natural to think of the outer part of the Earth in terms of layers (for instance, rock, soil, water, etc.), noting that these layers can have interfaces with complex geometry. Consider a layered material with each material layer I characterized by a distinct density α_I and moving material interfaces defined by equations of the form

$$F_i(x,y,z,t) = \eta_i(x,y,t) - z = 0 \tag{2}$$

where the subscript i refers to the material interface at the bottom of the Ith material layer and layer $I+1$ is vertically above layer I. A mass conservation equation for a given material layer in this system can be obtained by arbitrarily choosing, at an instant in time, a "disk" from the material layer of interest (Figure 1). The volume V of the disk is defined by fixed vertical sidewalls of area A_{side}, a bottom material interface of area Σ_i and velocity $\boldsymbol{v}_i$, and a top material interface of area Σ_{i+1} and velocity $\boldsymbol{v}_{i+1}$. The projected area of the material interfaces into the x-y plane is A_{xy} and the closed circumference of this area is S. Mass conservation for this disk can be written as

$$\frac{d}{dt}\int_V \alpha_I dV + \int_{A_{side}} \alpha_I \mathbf{u}\cdot\mathbf{n}_H dA - \int_{\Sigma_{i+1}} \alpha_I(\mathbf{u}-\mathbf{v}_{i+1}) \cdot\mathbf{n}_{i+1}dA + \int_{\Sigma_i}\alpha_I(\mathbf{u}-\mathbf{v}_i)\cdot\mathbf{n}_i\,dA - \int_V \Gamma dV = 0 \tag{3}$$

where $\mathbf{u} = (u, v, w)$ is the material velocity, $\mathbf{n}_H$, the outward pointing unit normal on the area A_{side}, is in the horizontal x-y plane, and Γ is a volume rate of mass production or destruction. The unit normal on the ith material interface $\mathbf{n}_i$ points from material layer I into material layer I-1 and from (2) can be calculated as

$$\mathbf{n}_i = \frac{\nabla F_i}{|\nabla F_i|} = \frac{[S_x, S_y, -1]}{\sqrt{S_x^2+S_y^2+1}}$$

where $S_x = \frac{\partial \eta_i}{\partial x}$ and $S_y = \frac{\partial \eta_i}{\partial y}$ are the material interface tangent slopes in the x and y directions respectively. We progress from (3) by noting that following the analysis presented by *Crank* [1984],

$$\mathbf{v}_i\cdot\mathbf{n}_i = -\frac{\partial \eta_i}{\partial t}\frac{1}{|\nabla F_i|} \tag{4}$$

(use the chain rule to take the total derivative of F in (2)) such that

$$\alpha_I(\mathbf{u}-\mathbf{v}_i)\cdot\mathbf{n}_i = \frac{\Omega_i}{|\nabla F_i|} \tag{5}$$

where

$$\Omega_i = \alpha_I\left[\frac{\partial \eta_i}{\partial t} + uS_x + vS_y - w\right] \tag{6}$$

Ω_i is the mass flux across interface i and by our sign convention is positive for flow from the Ith layer into the underlying (I-1)th layer. Further, an area element on Σ_i can be equated to an area element on the projected area A_{xy} through the relationship $dA_i = \sqrt{1+S_x^2+S_y^2}dA_{xy} = |\nabla F_i| dA_{xy}$ such that

$$\int_{\Sigma_i} \Psi dA = \int_{A_{xy}} \Psi|\nabla F_i| dA \tag{7}$$

where $\Psi(x, y)$ is an arbitrary function.

[7] Using (5) in (3) modifies the single-layer mass balance to

$$\frac{d}{dt}\int_V \alpha_I\, dV + \int_{A_{sides}} \alpha_I\mathbf{u}\cdot\mathbf{n}_H dA - \int_{\Sigma_{i+1}} \frac{\Omega_{i+1}}{|\nabla F_{i+1}|}dA + \int_{\Sigma_i}\frac{\Omega_i}{|\nabla F_i|}dA - \int_{V_i}\Gamma dV = 0 \tag{8}$$

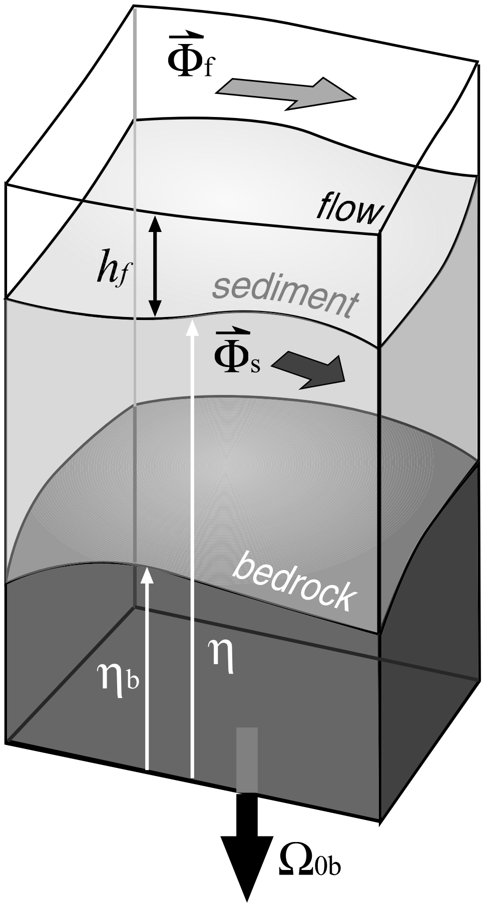

Figure 2. Definition sketch for a three-layer system: rock, sediment, and flow.

Application of the three dimensional generalization of the Leibnitz rule (sometimes referred to as the Reynolds transport theorem) moves the time derivative under the integral:

$$\int_V \frac{\partial}{\partial t}\alpha_I - \Gamma\, dV + \int_{A_{sides}} \alpha_I \mathbf{u}\cdot\mathbf{n}_H\, dA + \int_{\Sigma_{i+1}} -\alpha_I \mathbf{v}_{i+1}\cdot\mathbf{n}_{i+1} - \frac{\Omega_{i+1}}{|\nabla F_{i+1}|}\, dA + \int_{\Sigma_i} \alpha_I \mathbf{v}_i\cdot\mathbf{n}_i + \frac{\Omega_i}{|\nabla F_i|}\, dA = 0 \quad (9)$$

or, using the relationship in (4)

$$\int_V \frac{\partial}{\partial t}\alpha_I - \Gamma dV + \int_{A_{sides}} \alpha_I \mathbf{u}\cdot\mathbf{n}_H dA + \int_{\Sigma_{i+1}} \frac{\alpha_I}{|\nabla F_{i+1}|}\frac{\partial \eta_{i+1}}{\partial t} - \frac{\Omega_{i+1}}{|\nabla F_{i+1}|} dA + \int_{\Sigma_i} -\frac{\alpha_I}{|\nabla F_i|}\frac{\partial \eta_i}{\partial t} + \frac{\Omega_i}{|\nabla F_i|} dA = 0 \quad (10)$$

The relationship (7) can be used to write (10) in terms of integrals over the projected area

$$\int_{A_{xy}} \int_{\eta_i}^{\eta_{i+1}} \frac{\partial}{\partial t}\alpha_I - \Gamma dz dA + \int_S \int_{\eta_i}^{\eta_{i+1}} \alpha_I \mathbf{u} dz\cdot\mathbf{n}_H dS + \int_{A_{xy}} \alpha_I \frac{\partial \eta_{i+1}}{\partial t} - \Omega_{i+1} dA + \int_{A_{xy}} -\alpha_I \frac{\partial \eta_i}{\partial t} + \Omega_i dA = 0 \quad (11)$$

Applying the divergence theorem to the contour integral around S reduces all terms in (11) to integrals over the projected area A_{xy}

$$\int_{A_{xy}} \int_{\eta_i}^{\eta_{i+1}} \left(\frac{\partial}{\partial t}\alpha_I - \Gamma\right) dz\, dA + \int_{A_{xy}} \nabla_H\cdot\Phi\, dA + \int_{A_{xy}} \left(\alpha_I \frac{\partial \eta_{i+1}}{\partial t} - \Omega_{i+1}\right) dA + \int_{A_{xy}} \left(-\alpha_I \frac{\partial \eta_i}{\partial t} + \Omega_i\right) dA = 0 \quad (12)$$

where $\nabla_H = [\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0]$ and $\Phi = \int_{\eta_i}^{\eta_{i+1}} \alpha_I \mathbf{u} dz$ is a line flux. Since the area A_{xy} is arbitrary the sum of arguments in (12) has to be identically zero, leading to

$$\int_{\eta_i}^{\eta_{i+1}} \frac{\partial}{\partial t}\alpha_I\, dz + \nabla_H\cdot\Phi + \alpha_I \frac{\partial \eta_{i+1}}{\partial t} - \alpha_I \frac{\partial \eta_i}{\partial t} - \Omega_{i+1} + \Omega_i - \int_{\eta_i}^{\eta_{i+1}} \Gamma dz = 0 \quad (13a)$$

which using the conventional Leibnitz rule is equivalent to

$$\frac{\partial}{\partial t}\int_{\eta_i}^{\eta_{i+1}} \alpha_I\, dz + \nabla_H\cdot\Phi - \Omega_{i+1} + \Omega_i - \int_{\eta_i}^{\eta_{i+1}} \Gamma dz = 0 \quad (13b)$$

Both forms of this conservation equation for the Ith layer will be the basis for developing a complete coupled transport equation for surface evolution.

[8] To apply (13), we divide the Earth's outer region into three domains (Figure 2).

[9] 1. The first is flow, which for most purposes means the region in which volumetric sediment concentrations $c = \alpha/\rho_s$, where ρ_s is sediment particle density, are $\ll 1$. The flow involves a fluid medium, typically water, air, or ice. For mass flows, for which sediment concentrations are of order 1, the flow is distinguished from the bed by its rate of motion and/or of shear.

[10] 2. The second is the quasi-static particulate material (the bed), in which sediment concentrations c are of order 1 and motion is slow relative to layer 1. The bed typically comprises sediment, saprolite, and/or soil. This bed layer need not be immobile but following convention we restrict motion of the soil/sediment layer to creep, i.e., relatively slow motion, and consider rapidly deforming material to be part of the flow.

[11] 3. The third is rock, which for purposes of this analysis is assumed to have fixed density and not to gain or lose mass to chemical reactions. We adopt this convention just to avoid proliferation of terms in the final equation; the terms used to account for changes in density and mass gain or loss to dissolved phases apply just as well to layer 3 as to layers 1 and 2.

[12] Note that this grouping into layers is largely for convenience. The formulation developed above is generic

and could be applied to as many layers as needed for a given problem.

[13] To apply (13) to our three-layer system, we define the following additional variables: The elevation η written with no subscript refers to the conventional topographic elevation, i.e., the elevation of the surface of the Earth. This is the top of the sediment column unless no sediment is present, in which case it is the top of the bedrock. Subscript f refers to the flow (layer 1), s to the sediment (layer 2), and r to the bedrock basement beneath the sediment (layer 3). In particular, the elevation η_r is the elevation of the basement surface.

[14] Applying (13b) to the flow gives

$$\frac{\partial}{\partial t}\int_{\eta}^{\eta+h_f}\alpha_f dz+\vec{\nabla}_H\vec{\phi}_f+\Omega_{out}(\eta)-\Omega_{in}(\eta+h_f)-\int_{\eta}^{\eta+h_f}\Gamma_f dz=0 \tag{14}$$

where $\vec{\phi}_f$ is the horizontal vector mass sediment flux per unit width, comprising bed, suspended, and wash load. Note that we have accounted for the sign convention introduced after equation (6) in the two interlayer transfer terms in (14): outflow of material at the base of the flow η implies a positive value of Ω_{out}, and inflow of material at the top of the flow $\eta + h$ implies a positive value of Ω_{in}.

[15] Next we treat the sediment column. For this case, the meaning of the terms is clearer if we use (13a):

$$\int_{\eta_b}^{\eta}\frac{\partial\alpha_s}{\partial t}dz+\alpha_s(\eta)\frac{\partial\eta}{\partial t}-\alpha_s(\eta_r)\frac{\partial\eta_r}{\partial t}+\vec{\nabla}_H\vec{\phi}_s+\Omega_{out}(\eta_r)-\Omega_{in}(\eta)-\int_{\eta_r}^{\eta}\Gamma_s dz=0 \tag{15}$$

[16] The terms in this equation represent, from left to right: changes in the sediment density through the column (e.g., compaction, or inflation of the sediment due to chemical changes); the rate of movement of the bed surface η; the rate of movement of the bedrock interface η_r; net flow of soil or sediment into or out of the column due to creep in the *x-y* plane; net sediment outflow across the sediment-bedrock interface; net sediment inflow across the sediment-flow interface; and production or dissolution of particulate mass within the sediment column. It is critical to distinguish between the terms involving movement of the interfaces (terms 2 and 3 in equation (15)) and the terms involving transfer of mass across the interfaces (terms 5 and 6 in equation (15)).

[17] Finally, the equation for the bedrock is similar except that by assumption the terms involving change of density and production or loss of mass from solution are assumed to be zero for the bedrock. Also, and again only for the sake of clarity, we drop the terms representing lateral flow of bedrock (the second term in both forms of (13)). A mass conservation equation that retains these terms is developed by *Mudd and Furbish* [2004], who point out the potential importance of these terms in cases like folding where significant horizontal bedrock motion occurs near the surface. The lower boundary of the bedrock is taken to be a fixed surface across which passes a mass flux of rock Ω_{0r}. Together these conditions give

$$\alpha_r(\eta_r)\frac{\partial\eta_r}{\partial t}+\Omega_{0r}-\Omega_{in}(\eta_r)=0 \tag{16}$$

[18] Note that, maintaining the sign convention introduced with equation (6), for tectonic input of rock (uplift of rock in the terminology of *England and Molnar* [1990]), $\Omega_{0r} = -\alpha_r w_t$ where w_t is the upward speed of the rock mass. The negative sign reflects the fact that uplift represents inflow of rock to the system; our sign convention is that a positive Ω across the base of any layer represents outflow from the layer. Note also that for present purposes, "tectonic" means any imposed vertical rock motion, including isostatic.

[19] The last step is to assemble these three equations into a generalized mass balance equation. The coupling between equations occurs through the mass transfer terms. Using the middle (sediment) layer as a reference we have

$$\int_{\eta_b}^{\eta}\frac{\partial\alpha_s}{\partial t}dz+\alpha_s(\eta)\frac{\partial\eta}{\partial t}-\alpha_s(\eta_r)\frac{\partial\eta_r}{\partial t}+\vec{\nabla}_H\vec{\phi}_s+\left[\alpha_r(\eta_r)\frac{\partial\eta_r}{\partial t}+\Omega_{0r}\right]+\left\{\frac{\partial}{\partial t}\int_{\eta}^{\eta+h_f}\alpha_f dz+\vec{\nabla}_H\vec{\phi}_f-\Omega_{in}(\eta+h_f)-\int_{\eta}^{\eta+h_f}\Gamma_f dz\right\}-\int_{\eta_r}^{\eta}\Gamma_s dz=0 \tag{17a}$$

rearranging terms and assuming α_r is constant gives

$$\Omega_{0r}+(\alpha_r-\alpha_s(\eta_r))\frac{\partial\eta_r}{\partial t}+\int_{\eta_r}^{\eta}\frac{\partial\alpha_s}{\partial t}dz+\vec{\nabla}_H\vec{\phi}_s-\int_{\eta_r}^{\eta}\Gamma_s dz+\alpha_s(\eta)\frac{\partial\eta}{\partial t}+\frac{\partial}{\partial t}\int_{\eta}^{\eta+h_f}\alpha_f dz+\vec{\nabla}_H\vec{\phi}_f-\Omega_{in}(\eta+h_f)-\int_{\eta}^{\eta+h_f}\Gamma_f dz=0 \tag{17b}$$

The terms in the square braces and curly brackets in (17a) come from equations (16) and (14) respectively. From left to right, the physical meanings of the terms in (17b) are (1) vertical rate of input or withdrawal of rock from the basement, which is positive for removal (subsidence), negative for input (uplift); (2) vertical rate of rise or fall of the basement-sediment interface (the top of the basement), multiplied by the density change across the interface; (3) compaction or dilation of the sediment column, which accounts, for example, for compactional subsidence; (4) horizontal divergence of particle flux within the sediment layer (e.g., by soil creep), (5) rate of creation or destruction (e.g., through geochemical processes) of particulate mass in the sediment column; (6) vertical rate of rise or fall of the sediment (land) surface; (7) temporal rate of gain or loss of particulate mass within the flow; (8) horizontal divergence of particle flux within the flow; (9) gain or loss of sediment through the top of the flow; (10) rate of creation or destruction of particulate mass within the flow.

[20] For the sake of clarity, we have explicitly treated only the conservation of particulate mass. Mass exchange with dissolved phases is handled through the two creation/destruction terms (5 and 10). A complete dissolved phase mass balance could be developed in parallel with the development of (17), except that the flux and storage terms (e.g., 4, 7, 8) would represent chemical rather than particulate components.

4. Scaling the Mass Balance Equation

[21] It is highly unlikely that for any real case all the terms in (17) would be of comparable magnitude. Thus the first step in applying (17) is to specialize it to the problem at hand by dropping unimportant terms. This can be done by inspection or by formally scaling the terms. To scale the equation, we estimate the order of magnitude of each term by replacing the variables in it by representative values. Derivatives are replaced with algebraic expressions so that, for instance, $\partial\phi/\partial x$ becomes ϕ_0/L where ϕ_0 is a representative value of ϕ and L is a distance over which ϕ can be expected to change by a significant fraction of its representative value. Ratios of these representative values are nondimensional numbers, analogous to the familiar Reynolds and Froude numbers. They can be used to divide the domain of the equation into regimes governed by specific combinations of terms. Equation (17) has enough possible combinations of terms that it would be tedious to do this for the whole equation. Instead we give an example to show how the process works.

[22] As noted above, terms 7 and 8 represent, respectively, time changes in sediment storage in the flow and spatial changes in sediment flux. *Kneller and Branney* [1995] have emphasized the relative importance of temporal versus spatial changes in controlling clastic deposition; they focus on turbidites but their approach can be applied to any instance of deposition from physical transport. They present a system based on a phase plane for deposition which essentially classifies flows based on the signs and relative magnitudes of terms 7 and 8 in equation (17).

[23] Scaling analysis provides a way of estimating the approximate relative magnitudes of terms 7 and 8. With sediment mineral density ρ_s, and characteristic scale values $\mathbf{T}$, $\mathbf{H}$, $\mathbf{C}$, $\mathbf{q_s}$, and $\mathbf{L}$ for, respectively, time variation, flow height, volumetric sediment concentration in the flow, volumetric sediment flux per unit width of flow, and length scale of variation, the ratio of term 7 to term 8 is of the order

$$\frac{\partial}{\partial t}\int_{\eta}^{\eta+h_f}\alpha_f dz/\vec{\nabla}_H\vec{\phi}_f \approx \frac{\rho_s\mathbf{CH}}{\mathbf{T}}\Big/\frac{\rho_s\mathbf{q_s}}{\mathbf{L}} = \frac{\mathbf{CHL}}{\mathbf{Tq_s}} \tag{18a}$$

If the sediment flux is dominantly bed load, the characteristic sediment flux value $\mathbf{q_s}$ must be estimated as is. However, in such cases, sediment concentration averaged over the whole flow depth is likely to be negligible, because the bed load layer is only a few grain diameters thick. Temporal storage effects are thus nearly always negligible in bed load dominated situations. The question of spatial versus temporal effects becomes important where a significant part of the transport is via suspension. In that case, the sediment moves approximately with the flow velocity so the flux $\mathbf{q_s}$ is of the order $\mathbf{CUH}$. where $\mathbf{U}$ is a characteristic flow velocity The timescale $\mathbf{T}$ in this case is of the order $\mathbf{H/W_s}$ where $\mathbf{W_s}$ is a characteristic settling velocity. This gives an even simpler form:

$$\frac{\mathbf{CHL}}{\mathbf{Tq_s}} \approx \frac{\mathbf{L}}{\mathbf{H}}\frac{\mathbf{W_s}}{\mathbf{U}} \tag{18b}$$

For example, for a flow 10 m deep, a flow velocity of 1 m/s, with particles settling at 0.01 m/s being transported mainly in suspension, the velocity ratio in (18b) is of the order of 10^{-2}. In this case, temporal effects dominate over spatial effects if the spatial scale $\mathbf{L} \gg 1$ km.

5. Time Averaging

[24] The general mass balance equation (17) can in principle be applied to problems with timescales ranging from seconds to millions of years, so it is important to understand how it behaves under time averaging. We denote the average of any variable y over time interval T as $\bar{y} = (1/T)\int_{t_0}^{t_0+T} y(t)dt$. This is a linear operation, so the terms in (17) that do not contain time derivatives are simply replaced with their time averages. We assume that α_r, $\alpha_s(\eta_r)$, and $\alpha_s(\eta)$ do not change with time. The time derivatives are replaced by differences (equivalent to time-averaged vertical rates) to yield:

$$\begin{aligned}
&\overline{\Omega}_{0r}r - \alpha_s(\eta_r))\left[\frac{\eta_r(t_0+T)-\eta_r(t_0)}{T}\right] \\
&\quad + \frac{1}{T}\int_{\eta_r}^{\eta}[\alpha_s(t_0+T)-\alpha_s(t_0)]dz + \vec{\nabla}_H\overline{\vec{\phi}}_s - \int_{\eta_r}^{\eta}\overline{\Gamma}_s dz \\
&\quad + \alpha_s(\eta)\left[\frac{\eta(t_0+T)-\eta(t_0)}{T}\right] \\
&\quad + \frac{1}{T}\int_{\eta}^{\eta+h_f}\left[\alpha_f(t_0+T)-\alpha_f(t_0)\right]dz + \vec{\nabla}_H\overline{\vec{\phi}}_f \\
&\quad - \overline{\Omega}_{in}\left(\eta+h_f\right) - \int_{\eta}^{\eta+h_f}\overline{\Gamma}_f dz = 0
\end{aligned} \tag{19}$$

The magnitudes of the various terms are affected differently by this time averaging. For sustained flow of rock into or out of the system at typical plate tectonic rates, the value of term 1 would not be affected much by time averaging (i.e., it is always small). On the other hand, the net deposition rate $\left[\frac{\eta(t_0+T)-\eta(t_0)}{T}\right]$ with $\eta(t_0 + T) > \eta(t_0)$ has been shown by *Sadler* [1981] to decrease strongly with averaging timescale T. (Note that it is currently not known if the same is true of erosion rates.) Long-term deposition rates are comparable in magnitude to tectonic subsidence rates, but observed short-term rates are typically orders of magnitude higher. Not surprisingly, then, subsidence and uplift are generally not included in short-term morphodynamics analyses but become important on long timescales.

[25] Term 7 in (19), related to gain or loss of sediment from the flow, falls off very rapidly as averaging timescale T increases. As pointed out originally by *Rubin and Hunter* [1982], the local flow column cannot hold enough particulate material to act as a sediment source for very long. So as T becomes large, this term tends to zero. This is true even where, based on the analysis in the previous section, one would expect deposition on single-event timescales to be dominated by temporal effects: in the long run, deposited clastic sediment is supplied from upstream, not from overhead. On the basis of the analysis in the previous section, the timescale condition for neglecting the temporal term relative to the spatial term is $\mathbf{T} \gg \frac{\mathbf{CHL}}{\mathbf{q_s}}$.

6. Examples

[26] By dropping and combining appropriate terms from (17) we can create equations that describe a variety of useful special cases for Earth surface evolution.

6.1. Standard Exner Equation

[27] The original Exner equation is a balance of terms 6 and 8 in (17b). As mentioned earlier, term 7 is often included to give

$$\alpha_s(\eta)\frac{\partial \eta}{\partial t} + \rho_s \frac{\partial}{\partial t}\int_{\eta}^{\eta+h_f} C_f dz + \vec{\nabla}_H \vec{\phi}_f = 0 \qquad (20)$$

Note that $\alpha_s(\eta)$ is the bulk density of the deposited sediment, and that for consistency with convention we have rewritten term 7 by assuming that changes in bulk density of the flow are due only to change in volumetric sediment concentration in the flow C_f. In typical morphodynamic analyses involving noncohesive sediment (e.g., gravel and sand), the transport terms (2 and 3 in (20)) are parameterized in terms of the shear stress, which sets the sediment-transporting capacity of the flow. The shear stress is related via the flow field to the evolving topography (term 1). Cohesive sediment is discussed in section 6.5.

6.2. Sediment Precipitation and Sedimentation

[28] Globally, carbonate sediments represent the most important case of creation of particulate mass in the flow, but many evaporite sediments form in a similar manner. A simple short-term mass balance for sediment that precipitates in the water column involves (from left to right) changes in bed elevation, buildup of particulates in the water column, and precipitation/dissolution:

$$\alpha_s(\eta)\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial t}\int_{\eta}^{\eta+h_f} \alpha_f dz - \int_{\eta}^{\eta+h_f} \Gamma_f dz = 0 \qquad (21)$$

In the absence of lateral transport, the material produced is either stored in the water column (term 2) or deposited (term 1). Physical transport of carbonate sediments would generally involve the flux divergence term $\vec{\nabla}_H \vec{\phi}_f$, just as for clastic sediments. Particulate production is reflected by positive values of the production term Γ. Mass could also be precipitated or dissolved within the deposit, which would require adding term 5 from (17b).

6.3. Basin Subsidence

[29] On the basis of our inflow/outflow sign convention, the vertical rock velocity term is positive for subsidence (outflow). Hence with subsidence velocity σ we have $\Omega_{0r} = \alpha_r\sigma$. In the simplest case, the basement velocity is the negative subsidence velocity, i.e., $\frac{\partial \eta_r}{\partial t} = -\sigma$. (The exception is that, if one allows for conversion of sediment to basement through burial metamorphism, the basement would sink at a rate slower than $-\sigma$.) Assuming that the basement elevation does track the subsidence, the term $\Omega_{0r} + (\alpha_r - \alpha_s(\eta_r))\frac{\partial \eta_r}{\partial t}$ in equation (17b) reduces to $\alpha_s(\eta_r)\sigma$, representing the tectonic component. Problems involving basement subsidence generally have timescales long enough to assume dominance of the spatial term 8 over the temporal term 7 in (17b). Assuming in addition that the sediment does not flow once deposited and ignoring dissolution and precipitation in the sediment and flow, we end up with

$$\alpha_s(\eta_r)\sigma + \int_{\eta_r}^{\eta} \frac{\partial \alpha_s}{\partial t} dz + \alpha_s(\eta)\frac{\partial \eta}{\partial t} + \vec{\nabla}_H \vec{\phi}_f = 0 \qquad (22)$$

The second term, which accounts for compactional subsidence, is usually smaller than the first (tectonic) subsidence term. Important exceptions include muddy, slowly subsiding systems like the modern Mississippi delta, as well as other cases where the sediment column is thick and mud-rich [*Reynolds et al.*, 1991].

[30] Equation (22) is close to the usual statement of sediment mass conservation used to construct basin-filling models [*Paola*, 2000]. The third term is the spatial gradient in time-averaged sediment supply, which has to diminish downstream to provide material for deposition. Terms 1 and 3 reflect subsidence and changes in mean surface elevation respectively, while 2 represents compactional subsidence. Analyses to date have not retained the small difference in weighting of the subsidence versus surface elevation terms indicated by (22). Although minor compared to the other uncertainties in long-term basin modeling, the effect is interesting: replacing dense, compacted sediment at the bottom of the sediment column requires extraction of more particulate material from the flow (the third term, which is the supply) than does adding the equivalent vertical length of light, uncompacted sediment at the top of the column.

6.4. Simple Uplift With Soil Formation

[31] One might think of tectonic uplift as just the opposite of tectonic subsidence, i.e., the tectonic term plays the same role as in the previous example but has opposite sign. However, the more usual natural case involves soil formation. This makes the uplift case more interesting, in that the basement-soil interface η_r is substantially uncoupled from the tectonic uplift velocity. These effects are accounted for in the first two terms of equation (17b). They can be most clearly understood by looking at the extreme case where soil is produced in place and there is no gain or loss to transport or dissolution and precipitation. Recalling that rock uplift at vertical velocity u represents a negative flux at the base of the section, $\Omega_{0r} = -u\alpha_r$, (17b) becomes

$$-u\alpha_r + (\alpha_r - \alpha_s(\eta_r))\frac{\partial \eta_r}{\partial t} + \alpha_s(\eta)\frac{\partial \eta}{\partial t} = 0 \qquad (23a)$$

Even if all the densities are specified, the evolution of the sediment (soil) and the basement surfaces cannot be determined independently from (23a) alone without an auxiliary relation specifying, for instance, the soil production rate [*Anderson*, 2002; *Heimsath et al.*, 1997]. Two interesting limiting solutions to (23a) are

$$\frac{\partial \eta_r}{\partial t} = 0, \text{i.e.} - u\alpha_r + \alpha_s(\eta)\frac{\partial \eta}{\partial t} = 0 \quad \text{or} \quad \frac{\partial \eta}{\partial t} = \frac{u\alpha_r}{\alpha_s(\eta)} \tag{23b}$$

$$\frac{\partial \eta_r}{\partial t} = \frac{\partial \eta}{\partial t} = u \tag{23c}$$

In the first case, the rock input is converted to soil across a static bedrock-soil interface as fast as it is supplied. The land surface rises faster than the rock input rate, assuming that the soil is less dense than the rock. This is reflected by the α_r/α_s term in (23b). In the second case, a static soil layer of constant thickness rises passively, so that both interfaces move upward at the speed u of the bedrock uplift. No net soil production occurs.

[32] A more general conservation equation adds to (23a) the terms representing flow of soil, sediment transport, and loss of particulates to weathering/dissolution in the soil column. The time-dependent flow term (term 7 in equation (17b)) is left out based on the discussion above; on the timescales on which bedrock and soil interface motion are relevant, the time term is expected to scale out. The general equation is thus

$$- \alpha_r u + (\alpha_r - \alpha_s(\eta_r))\frac{\partial \eta_r}{\partial t} + \vec{\nabla}_H \vec{\phi}_s - \int_{\eta_r}^{\eta} \Gamma_s dz + \alpha_s(\eta)\frac{\partial \eta}{\partial t} + \vec{\nabla}_H \vec{\phi}_f = 0 \tag{24}$$

Equation (24) can be reduced to the forms presented by *Dietrich et al.* [2003], *Anderson* [2002], and *Mudd and Furbish* [2004] by dropping terms, except that Mudd and Furbish also retain terms accounting for horizontal bedrock velocity.

6.5. Simple Diagenesis

[33] In the simplest case, diagenesis involves increasing the mass in the sediment column via precipitation with no change in the column length or interaction with the rock and flow layers. This leads to a balance between terms 3 and 5 in (17b), noting that the production term Γ_s is positive:

$$\int_{\eta_b}^{\eta} \frac{\partial \alpha_s}{\partial t} dz - \int_{\eta_b}^{\eta} \Gamma_s dz = 0 \tag{25}$$

6.6. Erosion of Fine-Grained and Cohesive Material

[34] Sediment transport in suspension introduces a lag (time or distance) between changes in bottom stress or velocity and deposition. Also, cohesive effects that typically arise in fine-grained material mean that entrainment of sediment into the flow typically requires a much higher shear stress than that for which settling and deposition can occur. Sediment transport can be thought of as increasingly less "reversible" as cohesive effects become dominant. For present purposes, bedrock can be considered an extreme form of cohesive material.

[35] In general the strategy for these cases is to consider erosion and deposition separately. This is especially clear for extremely cohesive sediments like clays, for which deposition and deposition may be separated by long distances, and bedrock, for which eroded material may include both dissolved and particulate fractions. The dissolved component is returned to the solid phase by precipitation, perhaps far away and in an unrelated environment. Ignoring tectonic effects, a simplified form of (17b) illustrates how the mass balance works in these cases, assuming the eroded surface to be bedrock:

$$\alpha_r \frac{\partial \eta_b}{\partial t} + \frac{\partial}{\partial t}\int_{\eta}^{\eta+h_f} \alpha_f dz + \vec{\nabla}_H \vec{\phi}_f - \int_{\eta}^{\eta+h_f} \Gamma_f dz = 0 \tag{26}$$

Here the first term represents erosion from the bed (and therefore negative). The particulate fraction of the eroded mass appears as temporal or spatial increase in sediment concentration or flux, (terms 2 and 3, respectively). The chemical fraction of the eroded mass is accounted for via term 4, which would be negative (particulate material is going into solution). In a mass balance for the dissolved phase, this term would change sign and the material would appear as increases in storage or flux terms analogous to terms 2 and 3 above.

7. Geomorphic Flux Laws

[36] As mentioned above, the Exner equation is often the starting point for geomorphic and stratigraphic models, but it is usually not the main focus. The main challenge has been modeling the sediment flux term $\vec{\nabla}_H \vec{\phi}_s$ that appears in long-term equations like (22) and (24). The approach taken to modeling the sediment flux changes as we move from human to planetary timescales. On short timescales, tectonic influences are negligible, slowly varying aspects of the topography (e.g., the mean topographic slope) can be assumed constant, and one generally has quite a bit of information about the flow (e.g., detailed hydrologic or meteorologic records). Thus it is feasible to calculate the flow in detail, find the sediment flux, and then let the bed evolve via the standard Exner equation (20).

[37] On planetary timescales, tectonic motion comes into play, and the sediment and bedrock surfaces evolve together. It seems both impractical and unwise to attempt to retain the full suite of hydraulic and other details at long timescales. Instead, the usual approach has been to subsume these details into pragmatic semiempirical flux laws. Frequently these flux laws "short-circuit" the sediment-fluid connection (embodied in the sediment and flow layers in equation (17b)) by relating the sediment flux directly to the topography. Two simple and commonly used forms are (leaving out basement dynamics)

Diffusion

$$\alpha_s(\eta)\frac{\partial \eta}{\partial t} = A\nabla_H^2(\eta) \tag{27a}$$

First-order wave

$$\alpha_s(\eta)\frac{\partial \eta}{\partial t} = \vec{A} \bullet \vec{\nabla}_H(\eta) \quad (27b)$$

These can be obtained directly from the Exner equation (20) by dropping the second (time-dependent) term and relating the flux to topography by

$$\vec{\phi}_s = -A\vec{\nabla}_H \eta$$

for diffusion

$$\vec{\phi}_s = -\vec{A}\eta$$

for a first-order wave. In both cases A is a constant that encapsulates the effects of the unresolved hydraulic and other processes. Discussions of techniques for developing long-term flux laws are provided in, among many others, *Paola* [2000] and *Dietrich et al.* [2003]. Dietrich et al. refer to these longer-term flux laws as ‘‘geomorphic flux laws’’, and propose several original forms for hillslope transport. Although exploration of this subject is just beginning, a likely outcome is a suite of transport laws that explicitly depend on timescale.

8. Conclusion

[38] Our focus here has been to develop a general mass balance equation for Earth surface dynamics that includes geologic processes usually left out of engineering-oriented treatments. This requires a careful accounting of mass flow rates into and out of the rock, sediment, and flow domains of the Earth's surface. Equation (17) in its two forms is a general Exner equation that can be applied to problems in geosciences involving processes like tectonic uplift and subsidence, soil flow, and dissolution and precipitation as well as conventional sediment transport. The equation can be applied at any timescale, but for treatment of most long-term morphodynamics problems, term 7 in equation (17b) is negligible.

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